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RESPONSE SURFACE DETERMINATIONS IN ESTABLISHING THE RELIABILITY OF ONE-SHOT ITEMS

by

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RESPONSE SURFACE DETERMINATIONS IN ESTABLISHING THE RELIABILITY OF ONE-SHOT ITEMS

Nicholas T. Wilburn

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Abstract

In the past no satisfactory methods have been available for determining the reliability of one-shot items where extremely high reliability is required. This has been detrimental to the development of one-shot, automatically activated batteries for missile applications. The lack of a satisfactory method, one which will permit determination and statistical analysis of the failure points of the battery with respect to the specified environmental stresses, has been compensated by reliability programs based on designing the batteries to meet environmental stresses with large safety factors.

The response surface determination (RSD) method is proposed for the determination of mean failure points and reliability tolerance limits for a battery design with respect to operational environmental conditions, thermal and dynamic. It provides for the analysis of the battery responses as a function of two or more environmental stresses acting simultaneously, thus affording information on the effects of interactions between forces on the battery performance. Emphasis is placed on providing a maximum of reliability prediction data with small test sample sizes.

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CONTENTS

ABSTRACT i

INTRODUCTION 1

DISCUSSION 2

The Response Surface Determination (RSD) Method 2
Experimental Procedure 10
Experiment 1 12
Analysis of Experiment 1 20
Derivation of Experimental Test Data 22
Sample Size Determining Experiments 25

CONCLUSIONS 36

REFERENCES 37

Figures

1. Test Variables 4
2. Responses at Point A 5
3. Test Matrix 6
4. Test Results 8
5. Response Surface 9
6. Reliability Boundary 11
7. Trend in $\bar{Y}$ - KS with Increasing N 14
8. Cumulative Frequency Plot 15
9. Example 1 Results 21
10. "A" Sample Cumulative Frequency Distribution Plots 26
11. "B" Sample Cumulative Frequency Distribution Plots 27
12. "C" Sample Cumulative Frequency Distribution Plots 28
13. "C" Sample Cumulative Frequency Distribution Plots 29
14. "A" Sample Reliability Boundaries, 1 Test per Design Point 31
15. "B" Sample Reliability Boundaries, 2 Tests per Design Point 32
16. "C" Sample Reliability Boundaries, 4 Tests per Design Point 34
17. Effect of Replication on R. B. Estimates from No. 7 Samples 35

Appendices

APPENDIX I, Experiment 1 38
APPENDIX II 49
A. Test Data for "A", "B" and "C" Experiments 49
B. Analysis of Responses at Test Point (-1, -1) 53
C. Testing the Assumption of Normal Distribution at (-1, -1) 54
D. Analysis of "A" Sample Data 56
E. Analysis of "B" Sample Data 62
F. Analysis of "C" Sample Data 67

RESPONSE SURFACE DETERMINATIONS IN ESTABLISHING THE RELIABILITY OF ONE-SHOT ITEMS

INTRODUCTION

This report deals with a new procedure for establishing the reliability of one-shot items under simulated operational environments where extremely high reliability is required. One-shot items are herein defined as components or equipments which are expended in use and which, prior to use, do not lend themselves to non-destructive checkout tests from which their probability of successful operation or reliability can be inferred. The military one-shot items of perhaps the greatest interest to USAELRDL at present are the automatically activated battery power supplies which have been developed for guidance and control functions in a wide range of missiles. High reliability standards have been established for these batteries since failure in any sense will destroy the effectiveness of the missile. Unfortunately no effective and practical method has been available to date to determine the probable failure rate of a battery design under simulated missile environmental conditions. This has made it necessary to use compromise measures to achieve high engineering confidence in the reliability of the battery such as designing it to meet environmental requirements with high safety factors beyond the specified requirements.

The environmental safety factor concept[1] has resulted in a commendable history of high reliability in the field for many missile batteries. However it has been challenged from many sources on the basis of unnecessarily increasing the weight, size and cost of the batteries. The concept involves proving the battery design by testing relatively small samples under each critical environmental condition where the g force for shock, vibration, acceleration, spin, etc. is held at some arbitrarily chosen multiple of the required dynamic g force, normally four times. Failure of the battery design in any manner under a 4X environmental force results in redesign until the 4X capability has been demonstrated. Although statistical statements cannot be made as to the probability of failure at the required g level, the concept does provide for a high engineering confidence in the design.

Other approaches to battery reliability have been attempted including statistical ones such as the evaluation of the variability of the battery performance parameters [service life, activation time, maximum voltage, minimum voltage under heavy load pulses, etc.] under bench conditions at the required temperatures or under the specified dynamic g forces or even, in one program, under 4X g forces. The statistical approach used is to assume a normal distribution for the battery responses and then to evaluate each battery test sample for the mean value and the standard deviation for each response. The number of standard deviation units e.g. between the mean service life and the required service life must exceed a specified reliability standard, or K factor[2], which is related to the sample size, the desired maximum failure rate, and the confidence level used in making the reliability statement. Any program of this type has certain basic disadvantages. No real insight is gained into the effects on the battery performance of the thermal and dynamic environmental forces, unless this is sidestepped by testing at the 4X levels. Even under this condition the testing is done with one environment at a time, thus revealing nothing as to the potentially significant effects of interactions between the environmental variables.

It was recognized for some time that the environmental variables have a major effect on the reliability of a battery and that some test method was required to study the battery performance as a function of these environments treated as continuous variables. In fact the goal was to apply the same type of statistics as discussed before to demonstrate that the mean failure point of a battery with respect to an environment would be at least K standard deviation units above the required stress level. A first step in this direction was made in 1961 by the development of a step-stress technique[3] which permitted these determinations. The method had certain basic limitations which prevented its wide-scale application. It assumed that the test responses were normally distributed. It provided for the study of only one environment at a time. Its use required that the battery be tested at stress levels high enough to induce failure. This proved possible with the thermal environments, high and low temperature, but it was rapidly shown that the method was not generally adaptable to dynamic environments since practical test equipment in many if not most cases could not induce failure.

The RSD method has been developed at USAELRDL during the past two years to overcome these disadvantages. Although it has not been applied as yet in an actual reliability program, it is believed capable of providing the desired environmental capability data for a given battery design. Determining the mean minus KS reliability tolerance limit should be possible without testing to failure. Within the limits of practical test equipment, the method should provide for the testing under two or more different environmental stresses acting simultaneously, thus affording some of the desired insight into potentially destructive interactions between environmental stresses. Instead of merely assuming normal distribution of responses, the method provides a test which, though not conclusive, at least provides assurance in applying normal distribution statistics in making reliability predictions.

DISCUSSION

The Response Surface Determination (RSD) Method

The basic techniques of the RSD method are based on conventional design of experiments procedures which are thoroughly discussed in the literature.[4] These procedures provide for the study of the response of an item as a function of two or more continuous variables acting simultaneously. They are normally used to provide information as to the settings of the variables to yield an optimum condition for the response.

Translated into battery testing terms, the continuous variables of interest are the thermal and dynamic environments to which the battery is exposed in the missile operation. The responses to be evaluated are the battery's critical performance parameters: activation time, service life or capacity to minimum voltage, and maximum and minimum voltages under continuous or pulsating loads. Each of these responses are evaluated separately from the same test data and individual reliability predictions are made for each response.

The conventional design of experiments procedures give an indication of the mean or average response, but they do not provide for tolerance intervals based on reliability standards. The literature only covers the simple case of tolerance intervals around a linear regression line involving one variable. There is no practical technique given for curved functions involving two or more variables. Straightforward techniques were therefore derived for this and these, together with the design of experiments procedures, constitute the RSD method. This report is concerned entirely with the study of the battery response as a function of only two continuous variables since present testing equipment for the batteries permits only two environments, one thermal and one dynamic, to be applied simultaneously. For one-shot items capable of testing under more than two environments simultaneously, or with the advent of more sophisticated test equipment, the techniques as given will have to be expanded. This is entirely possible and consistent with conventional design of experiments procedures, though beyond the scope of this report.

The RSD theory will be developed through a series of illustrations. The mechanics of the operations will be developed later through a series of examples. Figure 1 shows a three dimensional space generated by the Y or response axis (e.g., battery capacity), the X_1 axis (e.g., high temperature as a continuous variable) and the X_2 axis (e.g., vibration g force, also as a continuous variable). Point A, the intersection of the axes, represents the combined requirement point, e.g., 10 seconds capacity at a 10 g vibration force at 165°F. The objective is to determine that the battery has the desired degree of reliability with respect to capacity at point A and also to determine how far along the X axes the given reliability standards will be met, and also for any given combination of X_1 and X_2 above the requirement levels.

The first step before exploring the three dimensional space is to determine if the battery design will meet the reliability standards at point A (Figure 2). A sufficient sample is tested at point A, 10 g and 165°F, and the data is analyzed to determine the mean capacity $\bar{Y}$ and the standard deviation, or measure of variability of individual points around the mean. A test is made to see if it is reasonable to assume that the points are normally distributed. (The techniques for all of these procedures will be explained in detail in the later experimental section). If the assumption of a normal distribution can be made, a normal curve can be superimposed on the Y axis, laid off in standard deviation units, and the $\bar{Y}$ - KS capacity value can be determined, where K represents the reliability standards.[2] Any $\bar{Y}$ - KS value over 10 seconds indicates adequate reliability at point A.

As shown in Figure 3, the X_1X_2 test design is then laid out. One or two batteries are first run along the X_1 and X_2 axes to obtain an approximate idea of the mean failure temperature and g force levels. As will often prove the case, the upper limit of the vibration equipment capability will not induce a failure. In this case this limit, hereafter referred to as the test equipment capability or TEC, will form the X_2 upper boundary. Some temperature B will form the boundary along the X_1 axis. Points C and D are established at the midpoints of the A-B and A-TEC intervals. This is a 3^2 factorial design with nine test points, eight of which remain to be investigated. Any appropriate scale can be used along each axis as long as equal

Test Variables

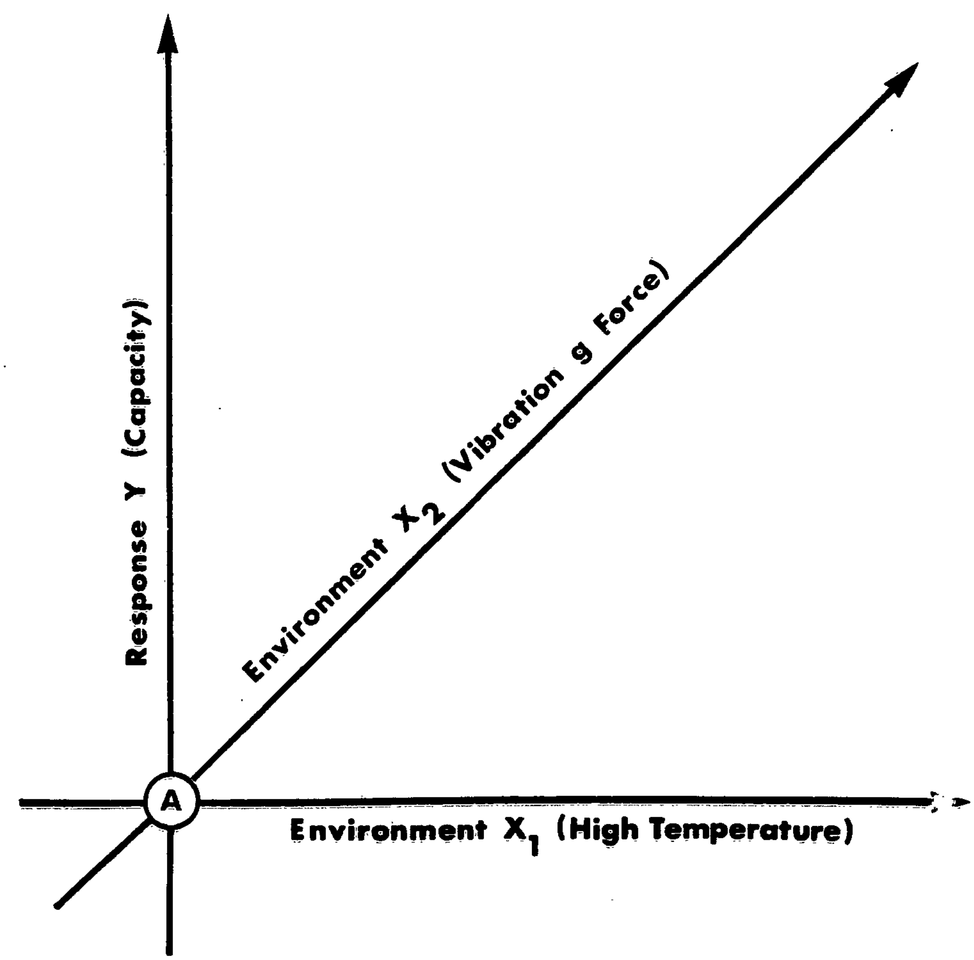


Figure 1

Responses at Point A

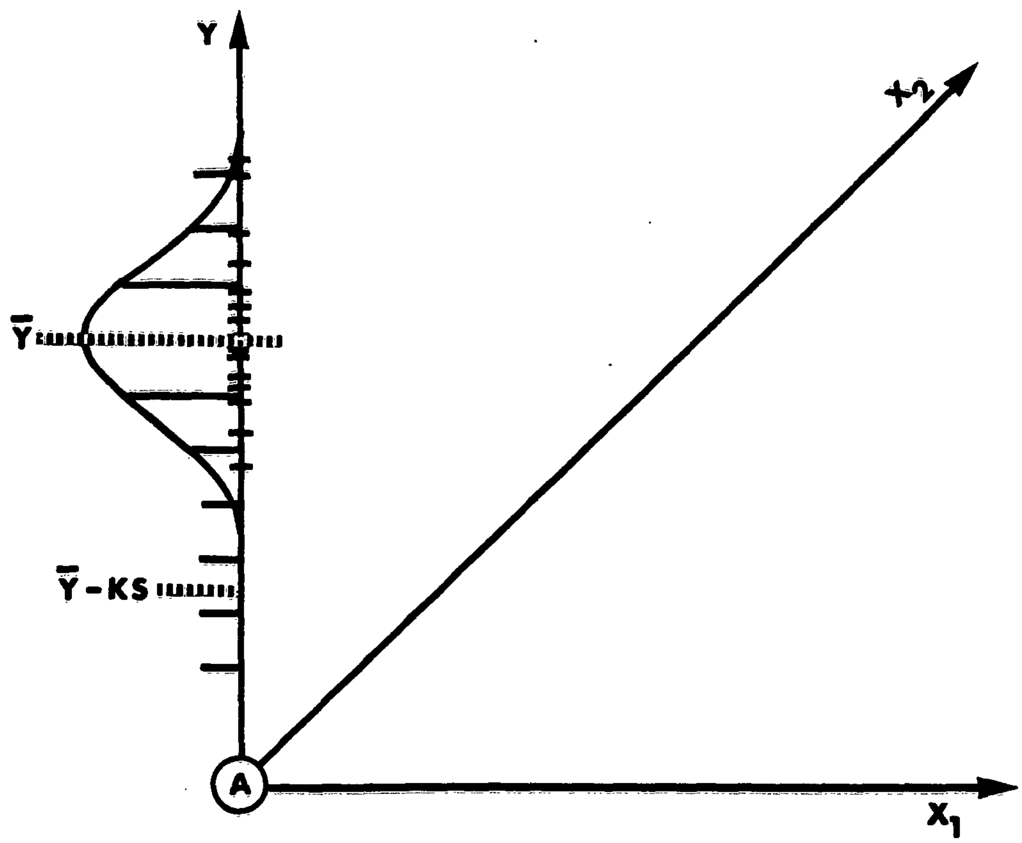


Figure 2

Test Matrix

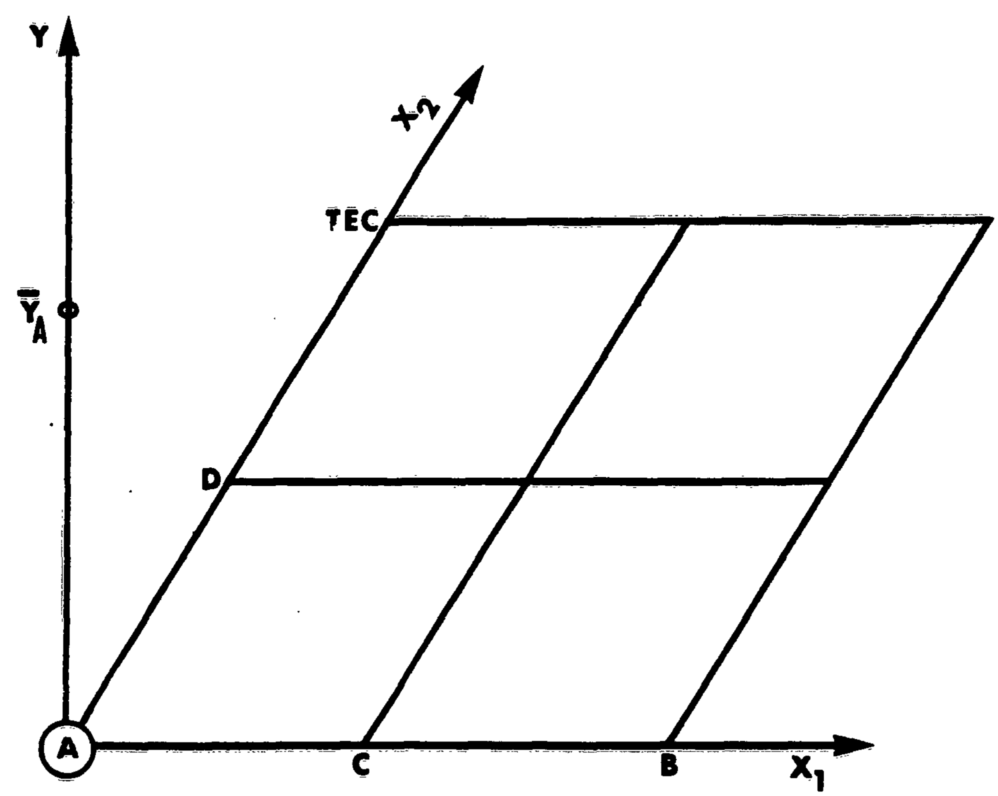


Figure 3

increments are held. In the case of vibration, assuming that the TEC is 40 g, the midpoint D can be 25 g with a linear scale or 20 g with a logarithmic scale, equal increments in both cases. Assuming B to be 205°F, C becomes 185°F.

Two tests are then conducted at each of the eight new test points giving capacities, Y responses, as shown on Figure 4. The test data points, as shown, may be considered typical of what might be expected if the battery response is influenced by the environments. Looking at the X_1 points at 10, 20 or 40 g the dropoff or regression of Y with respect to X_1 definitely does not appear to be linear but appears to suggest a mathematical relationship involving second order, quadratic, terms. Similarly with the X_2 points. The depression in the responses with increasing levels of X_1 gets more pronounced at higher levels of X_2. The same applies in the other direction. This is the interaction effect which means that the effect on the response of one variable cannot be stated properly without referring to the level of the other variable. Looking at the points as a whole it can be seen that a curved response surface can be fitted through the points starting with $\bar{Y}_A$ and dropping off with increasing levels of X_1 and X_2. This surface is shown on Figure 5.

The response surface does not fit uniformly through the test points but instead has a unique property. It is technically known as a least squares regression surface, which means that if the vertical deviations of the 18 test points (doubling $\bar{Y}_A$) from the surface are squared, then the sum of these squares will be at a minimum value for this one surface. It is based on a two variable, second order mathematical model of the form:

$$Y = b_0 + b_1X_1 + b_2X_2 + b_{11}X_1^2 + b_{22}X_2^2 + b_{12}X_1X_2$$

where the coefficients are derived from analysis of the 18 test points. The X_1X_2 term represents the interaction. The response surface is an estimate of the mean value of Y for any settings of X_1 and X_2 throughout the test area. Therefore the intersection of the surface with the 10 second Y plane is the mean failure contour which represents an important estimate of the population from which the battery sample was drawn. The estimate is that if additional samples were taken, half would pass and half would fail at any X_1 and X_2 settings along the contour. But this is of little value in predicting reliability of the battery design. What is required is another surface $\bar{Y}$ - KS below the response surface, the intersection of which with the 10 second plane will be the reliability boundary representing the highest values of settings of X_1 and X_2 at which the reliability standards can be met. A parallel surface drawn through $(\bar{Y} - KS)_A$ would not suffice since there is no reason to assume that this estimate of variance applies throughout the whole test region. Likewise it isn't known that the variance is uniform throughout and that the responses are normally distributed around the response surface throughout the whole test region.

An indication of a normal distribution had been demonstrated at the 10 g, 165°F point. A similar accumulation of data is not available elsewhere in the test region to make an additional test for normality. Therefore a basic assumption is made that if normality can be assumed at the combined requirement point, it can be assumed throughout the entire test region. It is

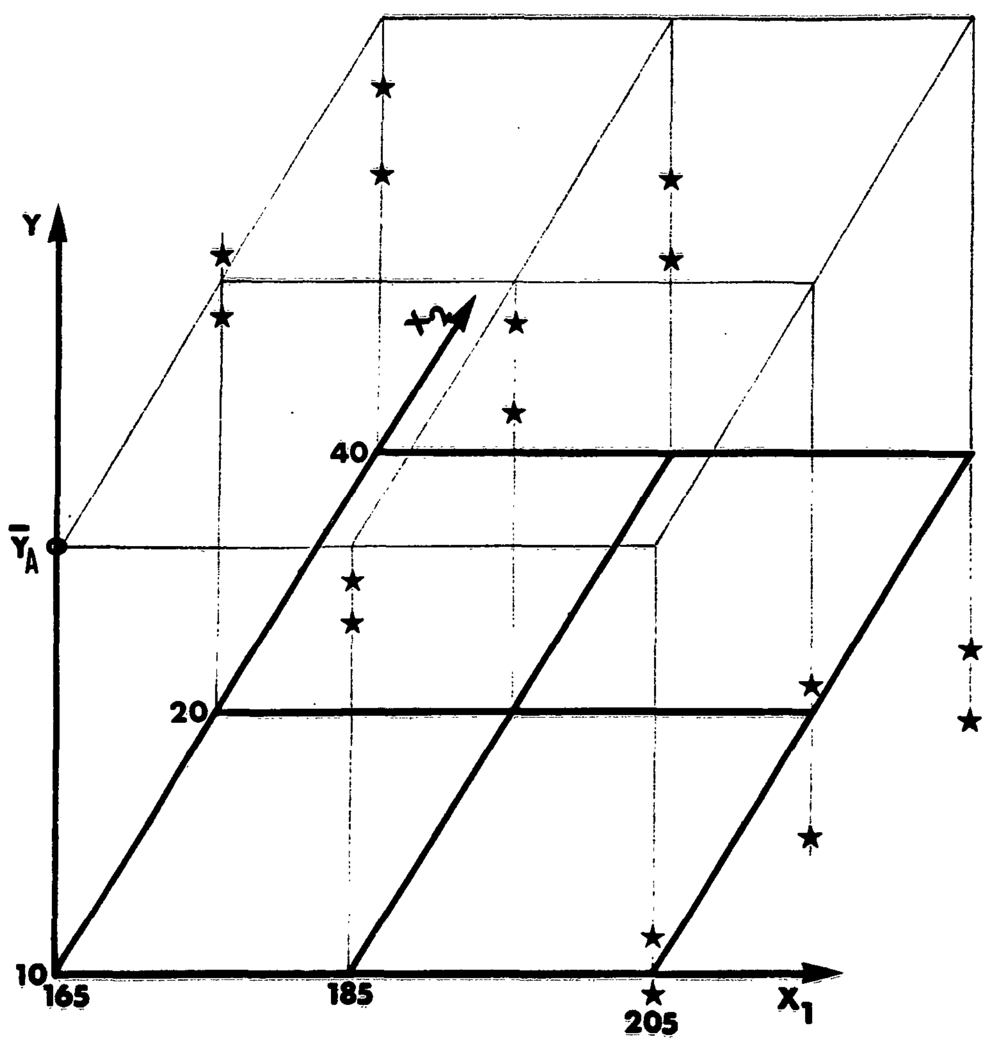


Figure 4

RESPONSE SURFACE

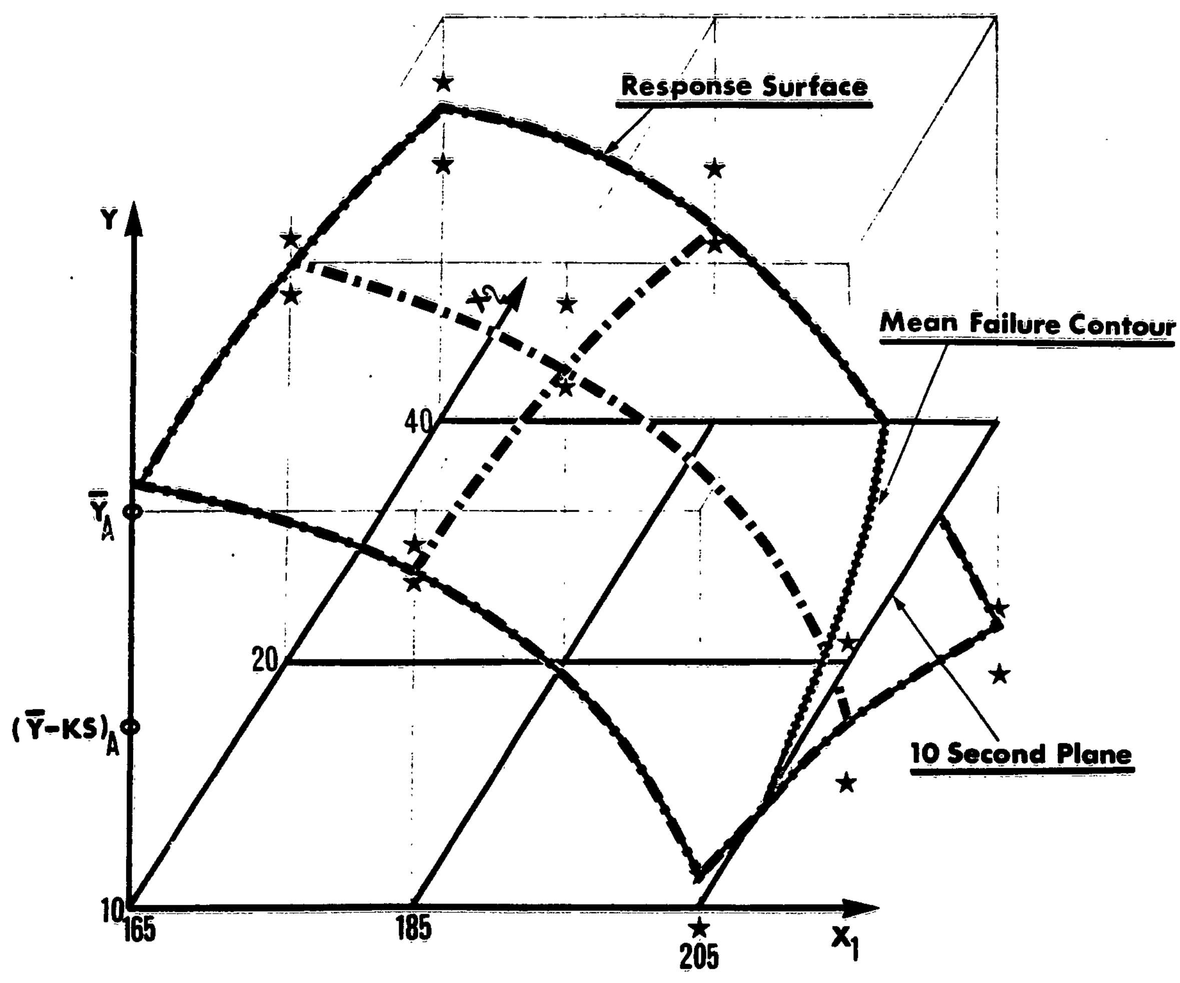


Figure 5

recognized that if departures from normality occur, they will influence the position of the reliability boundary, possibly weakening the value of these predictions. However they will not weaken the value of the prediction at the combined requirement point, the most important of the reliability predictions. The prediction as to how high we can go in temperature and still achieve reliable operation may be influenced slightly, but this is not as critical a prediction. The assumption of uniform variance can be tested for with the available data. This is done by determining the variance throughout the test area (except for the combined requirement point) and comparing it to the variance previously determined at the combined requirement point. If these two estimates of the variance cannot be proved to be significantly different, then uniform variance can be assumed by inference, at not too great a risk. If uniform variance can be assumed in this manner, the two estimates of variance can then be pooled to give a common estimate, based on all of the test data, of the variance and from this of the standard error throughout the entire test region. This latter estimate, multiplied by K, is then used to establish a $\bar{Y} - KS$ equation for a reliability boundary surface parallel to the response surface as shown in Figure 6.

The reliability boundary surface is drawn parallel to the response surface, $\bar{Y} - KS$ below it. The intersection with the 10 second plane is the required reliability boundary. The basic estimate now made from the sample is that the reliability standards will be met at any combination of X_1 and X_2 to the left of the reliability boundary including, of course, the combined requirement point of 165°F and 10 g. No statement should be made, by extrapolation of the reliability boundary, of how many g it would be possible to go to at 165°F and still meet the reliability standards. The only valid statement of this is, in excess of 40 g. The interrelation between the X and Y axes should be noted. Predictions can be made for different settings of Y as well. An 11 second plane could be added, for example, and the intersection of this plane with the reliability boundary surface would indicate the maximum settings of X_1 and X_2 at which 11 seconds of operation could be achieved in accordance with the same reliability standards. A limitless amount of reliability prediction data could actually be obtained once the reliability boundary surface has been determined, a further advantage of this approach to reliability.

The problem arises as to how positively the reliability predictions can be made. If there were no doubt at all that the responses were normally distributed about the response surface throughout the entire test region, and that the variance were completely uniform throughout the region, then there would be no hesitancy in stating that we are, for example, 95% confident in predicting that no more than one battery in a thousand will fail when tested at 193°F while undergoing a 10 g vibration. However the sample on which this estimate was based was also used to provide assurance in assuming normal distribution and uniform variance. It follows logically that the size of the sample must bear heavily on the faith in the prediction. Further comments will be made on this subject in the Conclusions section of this report once the mechanics of the RSD method have been explained and the area of sample size has been explored.

Experimental Procedure

The preceding section has attempted to explain the basic principles of the RSD method and general considerations governing its use in a practical

RELIABILITY BOUNDARY

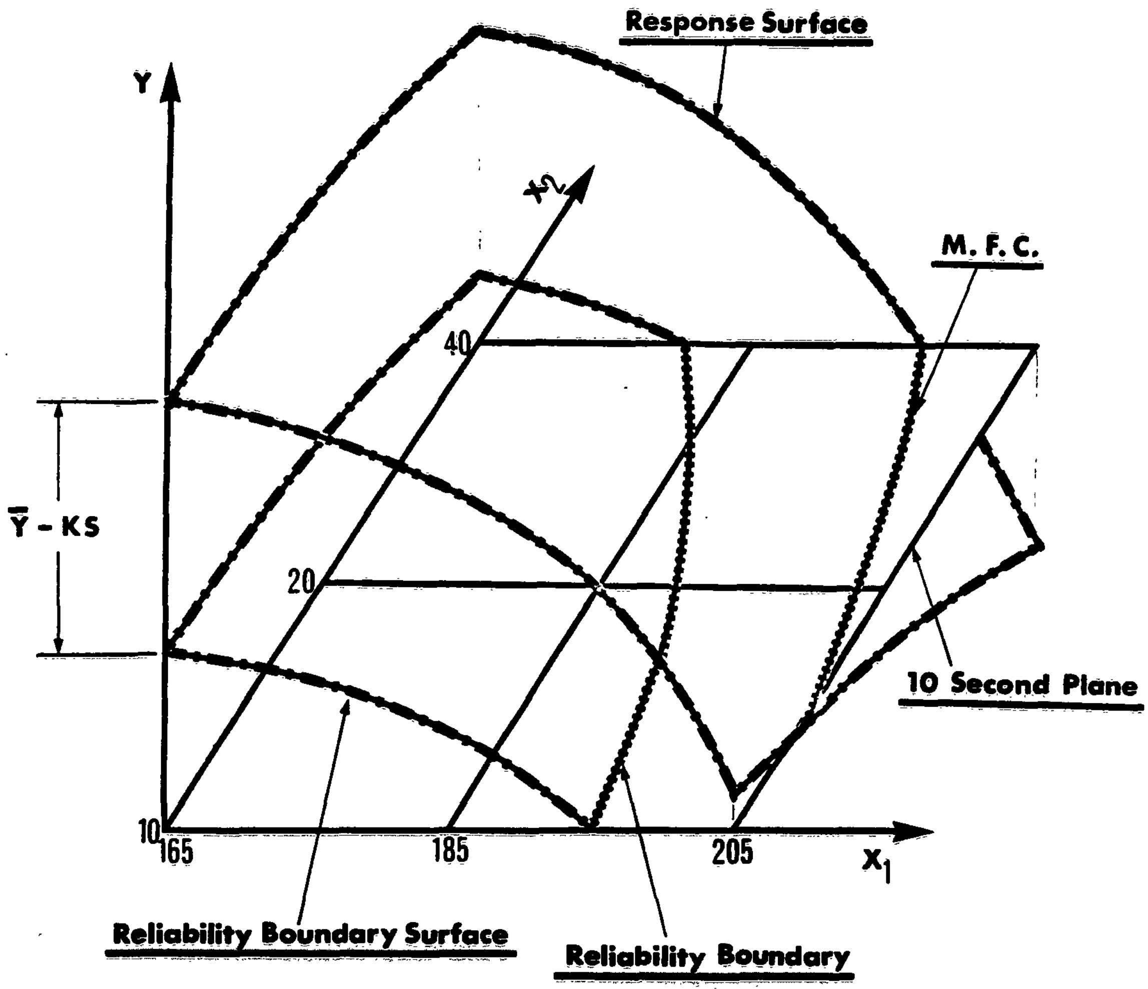


Figure 6

test program for determing the operational reliability of one-shot items such as battery power supplies for missiles. However, in this area, as in most in the field of applied statistics, no intuitive understanding is possible for the practicing engineer without detailed examples to study. Extensive examples have been prepared. These will be introduced by taking what is believed to be representative test data which will be treated, following the RSD method, in a step by step fashion through to the final reliability conclusions. Full details of all calculations are given in sequence in Appendix I.

Experiment 1

Among its other requirements, a battery is required to deliver a minimum of 10 seconds service life within specified voltage tolerances at a specified maximum temperature of 165°F, while being subjected to a sinusoidal vibrational force of 10g throughout the specified frequency range. It is desired to demonstrate that the battery design has a reliability, or probability of successful operation, of a least 99.9% expressed with a confidence level of 95%, i.e., admitting a chance of only one in 20 of stating that the battery has this reliability when, in fact, it does not. It is also desired to know the maximum temperature at which the battery will operate reliably when being vibrated at 10 g, and also the maximum g force to which the battery may be subjected while still maintaining 95% confidence in predicting that only one battery out of a thousand will give less than 10 seconds of operation when discharged at 165°F. Out of the total quantity of batteries available in the qualification test lot, it is desired to demonstrate the reliability with as small a test sample as possible to assure sufficient batteries being available for the study of other environmental variables which may prove to be more detrimental than the present high temperature-vibration combination. As a starting point, a small test sample of 15 is drawn at random from the lot.

Step 1. Seven batteries are selected for discharge test at the composite requirement point of 165°F (X_1) and 10 g (X_2), leaving eight for later exploration of the other eight points of the X_1, X_2 space of the 3^2 test design. The batteries are tested one at a time by stablizing at 165°F and then quickly removing them from the ambient, mounting them on the vibration equipment, activating and discharging them before any cooling effect takes place (or with appropriate temperature monitoring techniques to assure accurate temperature control). The batteries yield the following service times in seconds (Y responses) to minimum voltage:

15.74
15.63
14.41
16.61
15.47
14.51
18.79

Step 2. The results are analyzed (Appendix IA) for the mean capacity, 15.88 seconds, and the standard deviation, 1.488 seconds. The reliability defining K factor for a sample of size N=7, 99.9% reliability, and 95%

confidence level is 6.061[2]. Therefore, the critical $\bar{Y}$ - KS value is 15.88 - 6.061 x 1.488 = 6.86 seconds. This is well below the 10 second requirement. This indicates that either the battery design is basically unreliable or that perhaps the sample size was too small to allow the design, if good, to reveal itself as such. This latter view may be considered valid since the primary objective is to concede the unreliability, and to then search for reliability tolerance limits below the requirement levels, only when convinced that the reliability standards cannot be met. Evidence for such a decision may be based on trends in the $\bar{Y}$ - KS level with additional testing, up to a practical limit of, for example, 20 or 25 units. This approach will now be explored for the present example by testing additional randomly selected units one at a time and observing the trend in $\bar{Y}$ - KS with each sample as tested. The following results are obtained:

N	Y	$\bar{Y}$	K	S	KS	$\bar{Y}$ - KS
7		15.88	6.061	1.488	9.02	6.86
8	14.40	15.70	5.686	1.474	8.38	7.32
9	17.66	15.91	5.414	1.526	8.26	7.65
10	16.43	15.97	5.203	1.442	7.50	8.47
11	16.38	16.00	5.036	1.380	6.95	9.05
12	16.74	16.06	4.900	1.333	6.53	9.53
13	15.61	16.03	4.787	1.282	6.14	9.89
14	14.33	15.91	4.690	1.313	6.16	9.75
15	17.52	16.02	4.607	1.332	6.14	9.88
16	16.04	16.02	4.534	1.287	5.84	10.18
17	17.73	16.12	4.471	1.313	5.87	10.25

Graphically, the trend is shown in Figure 7. In this example, the trend is quite clearly upward due to a slight increase in $\bar{Y}$ and a definite decrease in S. The testing is stopped according to an arbitrary rule that two successive $\bar{Y}$ - KS values above the capacity requirement will establish the design reliability with an upward trend such as this. The reduction in K with increasing sample size is, of course, a factor in the increase of $\bar{Y}$ - KS, but not a significant one if the design actually had no chance of establishing itself. To indicate this, assume that $\bar{Y}$ and S remained at the N = 7 levels of 15.88 and 1.488. The increase in $\bar{Y}$ - KS with increasing N is as shown by the lower line which eventually reaches 10 seconds at N = 35. In practice, the existence of an upward trend, when testing up to a practical limit of 20 or 25 units, should be judged by an apparently significant difference in the two lines, as shown by the shaded area. With an inherently unreliable design, testing could be stopped presumably after a total N of 10 or 15. In any case, if the required $\bar{Y}$ - KS value is not reached by N = 20 or 25, testing is stopped and the response surface test design is then centered on the X_1, X_2 requirement point.

Step 3. Having judged the battery design to have adequate reliability at the X_1, X_2 requirement point, the next step is to make a determination of the approximate normality of the distribution of the individual test result points about the mean value of 16.12 for N = 17. This is done by plotting the cumulative distribution of the sample responses on arithmetical probability paper and observing if an approximate straight line is formed by the points, particularly in the range of P = 10 to P = 90. Calculations for the plotted points are given in Appendix IB. The plot is given in Figure 8.

Trend in $\bar{Y} - KS$ with Increasing N

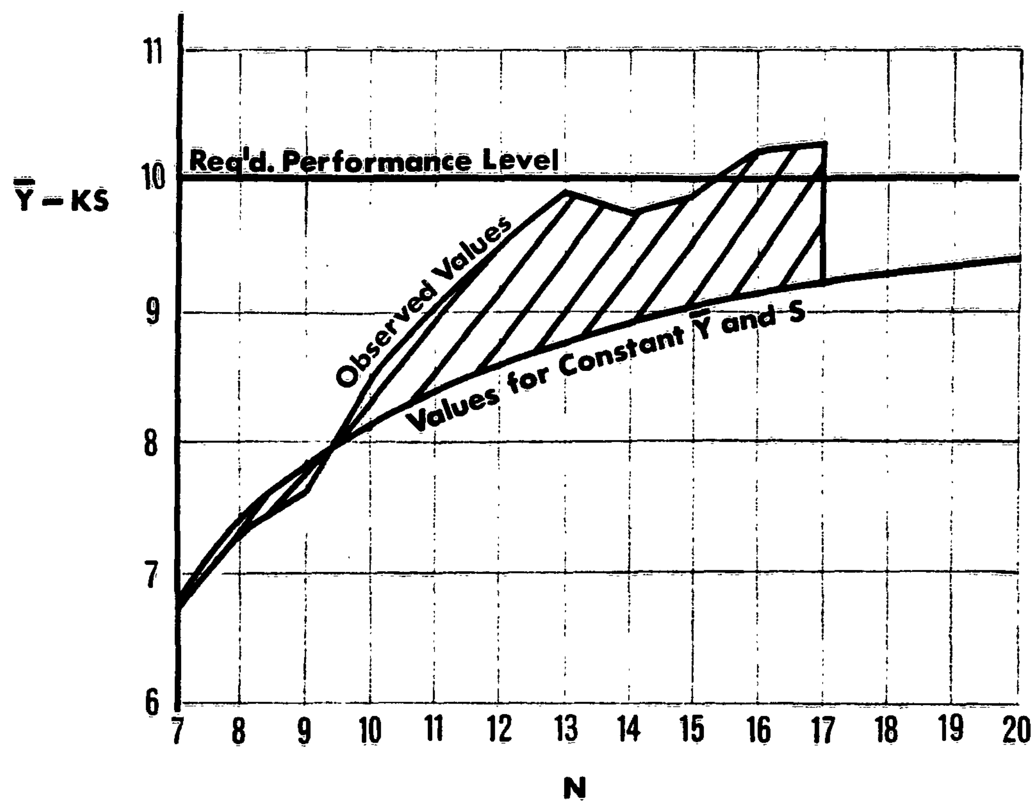


Figure 7

Cumulative Frequency Plot

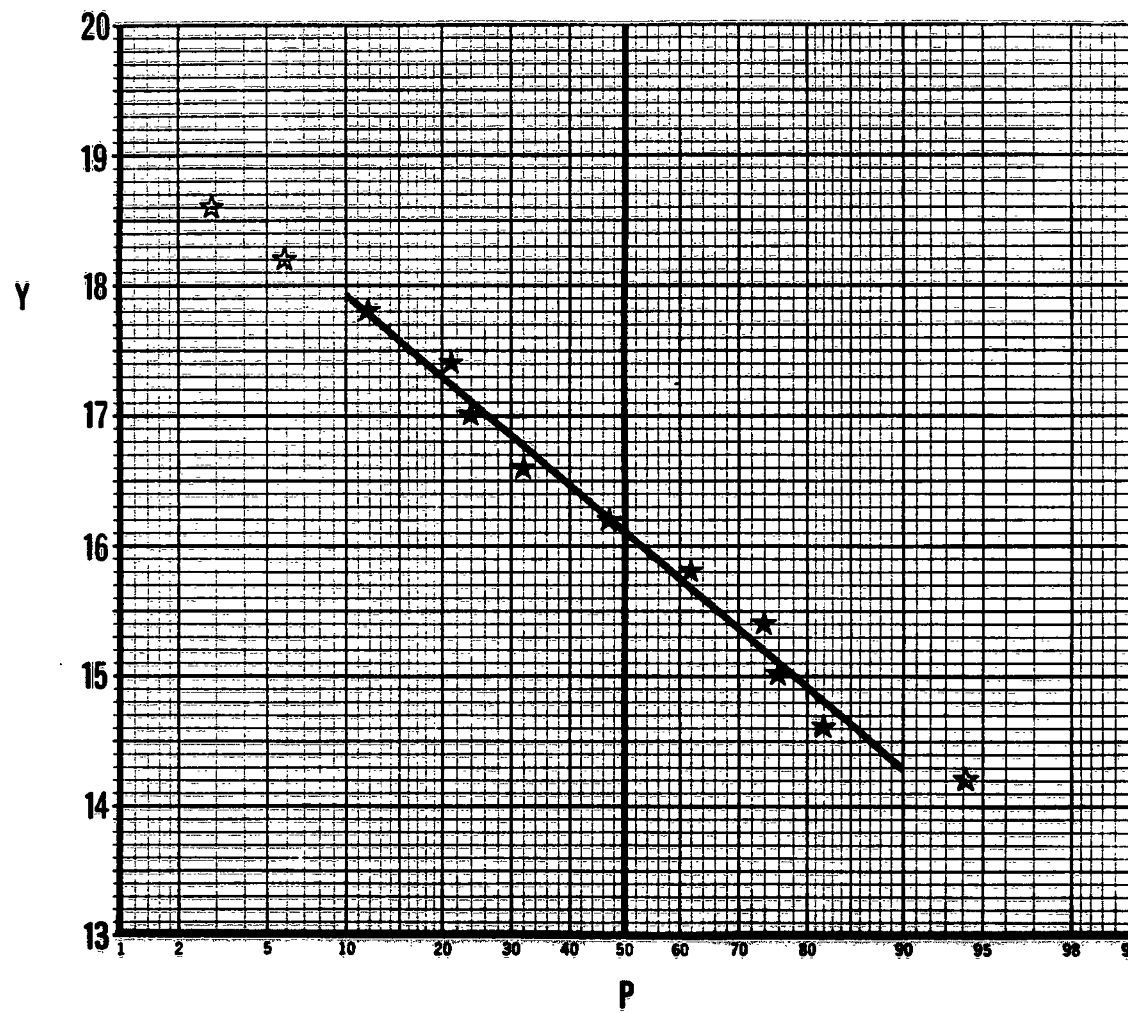


Figure 8

The points are noted to fall along a straight line very well. A straight line is the plot one obtains when plotting the cumulative distribution of a large or infinite sample from a known normally distributed population. Therefore, it is highly probable that the test sample was drawn from a normally distributed population. If the population were not normally distributed, in which case it would plot out as some curved line, then the probability is that the test sample plot would deviate from the straight line.

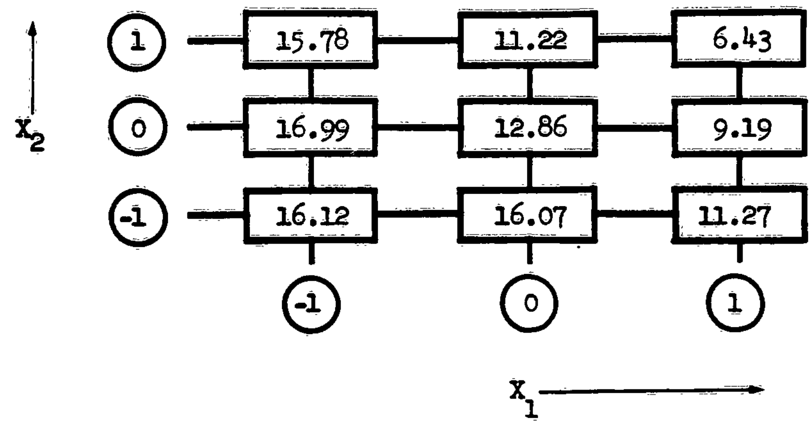
This test is regarded as an empirical indication of the suitability of assuming a normal distribution of individual Y responses about the mean value at the combined X_1, X_2 requirement point. Having satisfied the assumption at the most important point of the X_1, X_2 space, normality is now assumed throughout the entire test range of the X_1, X_2 space, recognizing, of course, that any departures from normality may have an influence on the eventual estimates of the battery reliability capabilities along the X_1 and X_2 axes, but not on the estimates in the region close to the X_1, X_2 requirement point.

Step 4. Having demonstrated reasons for assuming the reliability and the normality at the combined X_1, X_2 requirement point, and having obtained an estimate, 16.12, of the population mean and an estimate, 1.725, of the population variance (S^2) at that point, the test design to explore the X_1, X_2 space will be established. This will be a 3^2 factorial design with its lower boundaries established by the X_1 and X_2 requirement levels. The upper boundaries will be established, as explained in the previous section, preferably at an estimate of the mean failure point but, where this lies above the TEC level for a given environment, then at the TEC level. In this example, let it be assumed that the mean failure point at X_2 = 10 g has been estimated at 205°F. Let it also be assumed that the mean failure point at 165°F is in excess of 40 g, the vibration TEC. Therefore, the test design is established with X_1 limits of 165°F and 205°F and X_2 limits of 10 g and 40 g. The intermediate X_1 value is 185°F. The intermediate X_2 value, instead of being $(40 + 10)/2 = 25$ g, is set at 20 g to illustrate the technique of transforming a variable, in this case, on a logarithmic basis in which case 10 to 20 and 20 to 40 g represent equal increments along X_2. Coded values are assigned to the X_1 and X_2 values as -1, 0 and 1 levels. These coded values are of the utmost importance for keeping calculations as simple as possible. All equations and expressions from now on in this example will be based on the coded values. Thus, the combined X_1, X_2 requirement points will be referred to as (-1, -1), the 185°F, 40 g point as (0,1), etc. Based on the above, the test design is as shown in Figure 9.

Step 5. Tests are now run at each of the eight points of the design other than (-1, -1). The results are as follows:

Design Point		Response, Y
X_1	X_2	
(0	-1)	16.07
(1	-1)	11.27
(-1	0)	16.99
(0	0)	12.86
(1	0)	9.19
(-1	1)	15.78
(0	1)	11.22
(1	1)	6.43

The responses, shown on the test design, are as follows, with the 16.12 mean value introduced for the (-1, -1) point:



The responses indicate potentially significant effects on the response with increasing stress levels of X_1 and X_2. The coefficients are obtained (Appendix IC) for the two dimensional, second order mathematical model for the response surface:

$$Y = 13.52 - 3.67X_1 - 1.67X_2 - 0.76X_1^2 - 0.20X_2^2 - 1.13X_1X_2$$

An analysis of variance (Appendix ID) is now conducted to test the equation. Since the tests were not replicated (repeated) at the eight test points, no measure of the experimental error can be made and therefore no measure of the "lack of fit" of the model to the data can be made. The only important test of the data that can be made is the determination of whether or not the second order coefficients, b_{11}, b_{22} and b_{12}, are significant. It is seen from the analysis of variance that it cannot be concluded that the quadratic components are significantly different from zero. Therefore it may be concluded that a linear model of the form

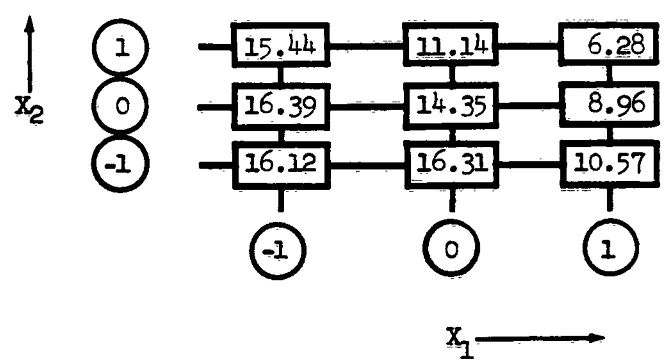
$$Y = b_0 + b_1X_1 + b_2X_2$$

would suffice. However in this case such a model is not fitted because

there is reason to doubt, for theoretical reasons, that the quadratic terms are not significant. Therefore the test design is replicated at the eight points (except for (-1, -1)) to check this as well as to check the experimental error and the "lack-of-fit" term. The additional test results, added to those previously determined, are:

Design Point X_1, X_2	Original Response, Y	Replicate Response, Y
(0, -1)	16.07	16.55
(1, -1)	11.27	9.87
(-1, 0)	16.99	15.79
(0, 0)	12.86	15.84
(1, 0)	9.19	8.72
(-1, 1)	15.78	15.09
(0, 1)	11.22	11.05
(1, 1)	6.43	6.12

The average responses, shown on the test design, are as follows with the 16.12 mean value again introduced for the (-1, -1) point.



The coefficients are obtained (Appendix IE) for the two dimensional, second order mathematical model based on the replicated test data:

$$Y = 14.33 - 3.70X_1 - 1.70X_2 - 1.65X_1^2 - 0.60X_2^2 - 0.92X_1X_2$$

An analysis of variance (Appendix IF) is now conducted to test for the significance of the quadratic components, b_{11}, b_{22} and b_{12}. Due to the existence of the extra test data, it is possible to divide the residual sum of squares into two parts, that due to experimental error and whatever is left which is called the lack of fit term, a measure of the inability of the mathematical model to fit the test data.

It is found that the quadratic terms are significant and that the lack-of-fit- term is not significant. Therefore the mathematical model is accepted as an adequate interpretation of the experimental data.

Step 6. It is now necessary to study the variance throughout the test space and to compare it to the variance at the (-1, -1) requirement point. The variance is determined (Appendix IG) by summing the squares of the differences between each of the 16 test results and the values predicted at the test points from the equation. The variance is found to be 1.00893, which is to be compared to the variance estimate of 1.72456 found for the 17 results at (-1, -1). The test region variance, 1.00893, is based on N - 5 = 11 degrees of freedom. The (-1, -1) variance was based on N - 1 = 17 -1 = 16 d.f. The $F_{11,\ 16}$ ratio = 1.00893/1.72456 = 0.59. This ratio falls within the limits of $F_{.95,\ (11,\ 16\ d.f.)}$ of 2.94 and of $\frac{1}{F_{.95,(16,11\ d.f.)}}$ of 0.302. These are the 95% level of significance limits of an F test[5] for determining acceptance or rejection of the hypothesis that there is no significant difference between the two estimates of variance. In this case the hypothesis is accepted that there is no significant difference between the variance at (-1, -1) and the variance throughout the remainder of the test area. It may therefore be concluded by inference that there is a uniform variance throughout the entire test region. The two estimates of variance are then pooled to give an estimate of S, the standard error throughout the entire test region. This value, from Appendix IH, is 1.197 seconds.

Step 7. The total sample size used in this exploration of the X_1X_2 space was 33, 17 at (-1, -1) and two at each of the other eight points. Of the starting 33 degrees of freedom, six were used in establishing the coefficients of the mathematical model, and the remaining 27 for establishing the pooled standard error. Twenty-seven is also the effective sample size in determining the K factor, which for N = 27, P = 0.999 and Confidence = 0.95 is 4.090. The KS factor is therefore 4.090 x 1.197 = 4.90 seconds. This factor is used in establishing the $\bar{Y}$ - KS reliability boundary value for any point in the X_1, X_2 test space. Actual equations may now be established for finding values at any settings in the three dimensional Y, X_1, X_2 space. Two of the most convenient equations are:

1. The equation for mean failure points of Y with respect to X_1 and X_2:

$$Y = 14.33 - 3.70X_1 - 1.70X_2 - 1.65X_1^2 - 0.60X_2^2 - 0.92X_1X_2 = 10 \text{ (Y requirement) or } 3.70X_1 + 1.70X_2 + 1.65X_1^2 + 0.60X_2^2 + 0.92X_1X_2 = 4.33$$

2. The equation for reliability boundary points of Y with respect to X_1 and X_2:

$$Y = 14.33 - 3.70X_2 - 1.65X_1^2 - 0.60X_2^2 - 0.92X_1X_2 = 10 + 4.90(KS) = 14.90 \text{ or } 3.70X_1 + 1.70X_2 + 1.65X_1^2 + 0.60X_2^2 + 0.92X_1X_2 = -0.57.$$

Substituting appropriate values of X_1 or X_2 (coded values, not actual ones) and solving the resulting quadratic equations (Appendix II and IJ)

values from which the mean failure contour (1) and the reliability boundary contour (2) are plotted in Figure 9. (The significance of curves (A) and (B) will be explained later).

Analysis of Experiment 1

The experiment is substantially completed with the plotting of the mean failure contour and the more significant reliability boundary. Much more could of course be done if necessary or desired. For example, if the reliability boundary had passed below the (-1, -1) requirement point (indicating a probability of more than one unit per thousand failing to give 10 seconds of service at (-1, -1), expressed with a 95% confidence level) then instead of launching a redesign of the battery, the missile user may decide that he will be satisfied if he can reliably get nine seconds of operation. An equation ($3.70X_1 + 1.70X_2 + 1.65X_1^2 + 0.60X_2^2 + 0.92X_1X_2 = -1.57$) will then permit determination of the nine second reliability boundary (intersection of the surface, which is $\bar{Y} - KS$ below the mean response surface, and the nine second plane). For a case of adequate reliability such as the present example, approximations of the X_1 and X_2 reliability boundaries can be made for higher requirement levels of Y, 11 or perhaps even 12 seconds. Single variable curves may be generated and studied, e. g. capacity as a function of temperature at a constant vibrational force anywhere from 10 to 40 g.

For the present purpose, however, it is now possible to make the required prediction of the battery design reliability at the temperature requirement of 165°F and at the vibration requirement of 10 g. The reliability standard, it is estimated, will be met anywhere to the left of the reliability boundary. Therefore the design has the desired reliability at the 165°F, 10 g point. If desired, it is possible to determine how many standard error units there are along the Y axis through (-1, -1) between the mean response surface and the 10 second plane and from this give an estimate of the actual reliability, 99.999 etc. However, this is of no practical significance, the main question being, has the reliability standard been met or not.

From the reliability boundary plot, estimates can be made of the battery design capability at environmental stress levels above the requirements. These estimates are qualified, of course, by the reliability standards. Thus, expressed with 95% confidence, it may be estimated that no more than one battery out of a thousand will fail to give at least 10 seconds of service under the following environmental conditions:

a. At a temperature of up to 188°F while undergoing a 10 g vibration

b. In excess of a 40 g vibrational force at a temperature of 165°F

c. At a temperature of up to 181°F while undergoing a 20 g vibration, etc.

Example 1 Results

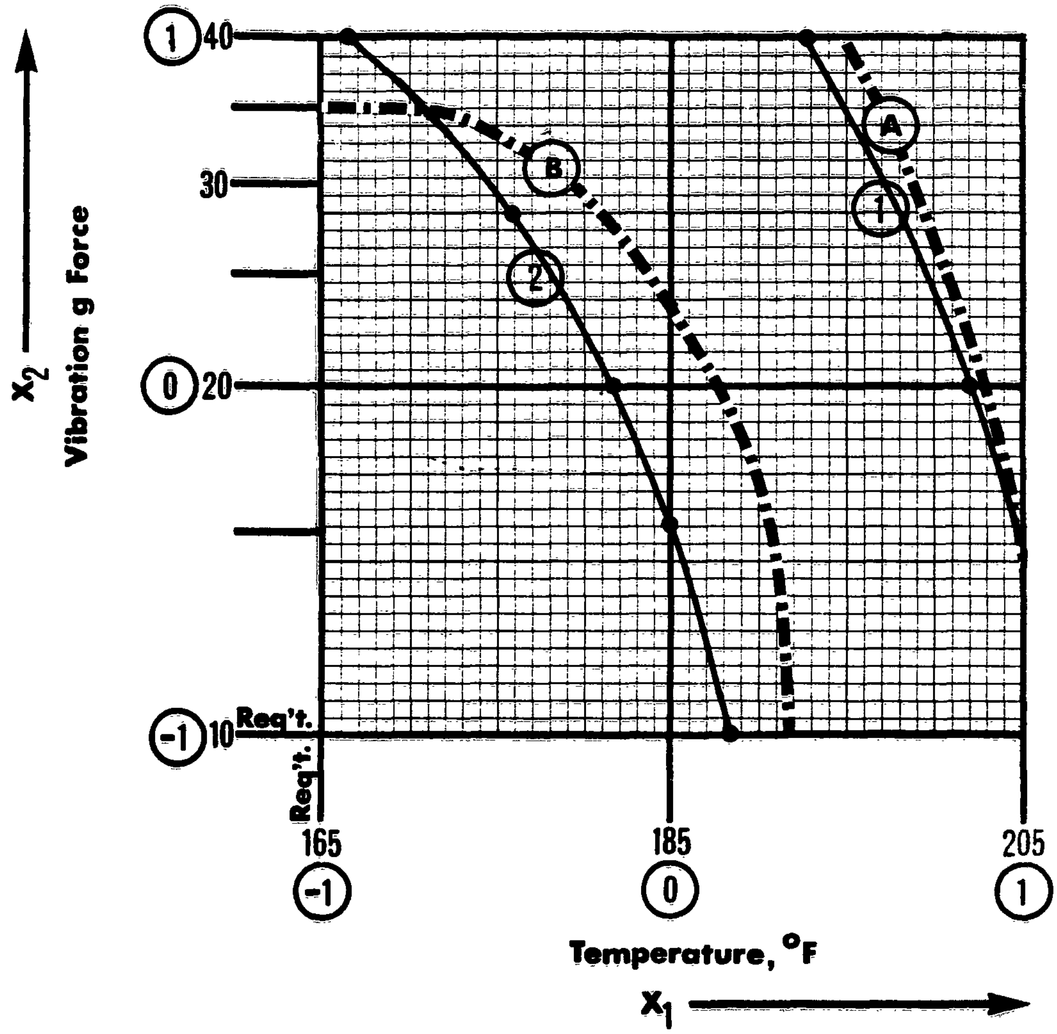


Figure 9

Derivation of Experimental Test Data

In the detailed example just completed, tests were conducted throughout the X_1, X_2 space, the data was analyzed and various estimates were made concerning the capabilities of the population from which the sample was drawn. These estimates in final form are represented by the reliability boundary contour. A logical question at this point would be, how close is this contour to the true picture? For example, how close is it to the average contour for an infinite number of samples of the same size drawn from the same population (in this case, future batteries produced exactly the same in every respect as the lot from which the test sample was drawn)?

There are two ways to answer such a question. One would be to produce enough batteries and run enough tests with samples of size $N = 30$ (14 at (-1, -1) and 2 at each of the other 8 points) to begin to understand the variability of the individual sample reliability boundaries. In this way, it could eventually be said whether or not the sample gave a "good" estimate of the true picture. This approach is not practical, of course, with expensive items and testing. The second way to answer the question is to know in advance the nature of the true picture, by conducting experiments with data which is derived from populations of known statistical parameters such as mean, standard deviation and, in this case, coefficients for the two variable second order regression, and the standard error. This will make it possible to conduct many relatively inexpensive experiments which can be made to closely simulate tests with actual one-shot items. The use of this technique has made it possible to thoroughly explore the RSD method in the vital area of the effect of sample size. Completely unexpected phenomena were observed in this manner, making it possible to derive methods for handling them.

In establishing the simulated test data, a hypothetical battery design was visualized with a mean capacity of 16.04 seconds and a standard deviation of 1.0 second at the (-1, -1) requirement point. The capacity Y of the battery was established as a function of X_1 (Temperature) and X_2 (Vibration g Force) in accordance with the following equation:

$$Y = 14.61 - 3.24X_1 - 1.69X_2 - 2.13X_1^2 - 0.94X_2^2 - 0.43X_1X_2$$

A variance, standard error squared, of 1.0 was selected for the X_1, X_2 test region. For $N = 30$ with 24 degrees of freedom for establishing the pooled standard error, the K factor is 4.17. Therefore, KS is 4.17, and a reliability boundary equation may be set up:

$$3.24X_1 + 1.69X_2 + 2.13X_1^2 + 0.94X_2^2 + 0.43X_1X_2 = 14.61 - 10.00 - 4.17 = 0.44$$

The contour generated from this equation is shown on Figure 9 as (B), the average reliability boundary for an infinite number of samples of size $N = 30$, d.f. = 24. It is by no means the reliability boundary for an infinite sample size. This limiting contour is generated from an equation similar to the above except that KS = 3.09, where the K factor of 3.09 is the limiting value of K for an infinite sample size with an infinite number of degrees of freedom. This infinite sample size contour, or universe reliability boundary, is discussed and presented later in Figure 17. Curve (A) of Figure 9 is the mean failure contour for any number of samples of any size. This contour is

generated by setting Y = 10 in the basic equation, giving:

$$3.24X_1 + 1.69X_2 + 2.13X_1^2 + 0.94X_2^2 + 0.43X_1X_2 = 14.61 - 10 = 4.61$$

It is seen on Figure 9 that the sample mean failure contour and reliability boundary obtained in Experiment 1 are relatively good approximations of the true values representing the known population. Just how good they are is a question that could best be answered following the experimental exploration of the effect of sample size in determining the response surfaces.

Curve (A) requires some comments. It passes through the X_1 axis just about at the limit of the test area. It passes through the X_2 axis at some point well beyond the test area. The coefficients of the equation which generates (A) were chosen deliberately so that the curve would have this general shape. This represents practical considerations in establishing the test area, as previously explained, with limits either at the estimated mean failure point or at the TEC (test equipment capability).

The method of obtaining actual simulated test data is as follows:

The base equation, $Y = 14.61 - 3.24X_1 - 1.69X_2 - 2.13X_1^2 - 0.94X_2^2 - 0.43X_1X_2$, will yield the following predicted values of Y when coded X_1 and X_2 values of -1, 0, or 1 are substituted:

X_2 \ X_1	-1	0	1
1	13.52	11.98	6.18
0	15.72	14.61	9.24
-1	16.04	15.36	10.42

A table of random normal numbers[5] with a universe mean of 0.00 and a universe standard deviation of 1.00 was entered at random in order to derive the test data. In Example 1, for example, column No. 38 was entered. The first 17 numbers are:

-0.30	-0.57	1.62	-0.43
-0.41	-1.53	0.39	-1.71
-1.63	2.75	0.34	1.48
0.57	-1.64	0.70	0.00
			1.69

These numbers added to 16.04 give the following Y responses at (-1, -1):

15.74	15.47	17.66	15.61
15.63	14.51	16.43	14.33
14.41	18.79	16.38	17.52
16.61	14.40	16.74	16.04
			17.73

These numbers have a mean value of 16.12 and a standard deviation of 1.313 which are estimates of the true population values of 16.04 and 1.000. With an actual large lot of batteries with a mean of 16.04 and and a standard deviation of 1.000, it is possible that a sample of $N = 17$ could have been drawn with a $\bar{Y}$ of 16.12 and an S of 1.313. In other words, the artificial populations can be as useful as actual populations in studying the effects of performance variability, actually much more so since all types of simulated test data in almost unlimited quantities can be rapidly and economically generated.

It was decided to explore the effects of sample size in using the RSD method by constructing samples with three representative sample sizes: $N = 15$, $N = 30$ and $N = 60$, with (-1, -1) point test quantities of 7, 14 and 28 respectively and with 1, 2 or 4 tests at each of the other eight points of the 3^2 test design. Ten examples were constructed for each of the three sample sizes for a total of 30 examples. The examples were constructed in such a way that the results of the analysis could be followed in stages for each of the 10 examples, i.e, from $N = 15$ to $N = 30$ and $N = 60$ with each increase based on additions to the previous data.

A table of random numbers[5] was entered to give 10 consecutive numbers of 61, 46, 10, 24, 85, 40, 38, 28, 58 and 17. These numbers were used to determine the column numbers to be used from the previously referenced table of random normal numbers. Thus the three No. 1 samples, 1A ($N = 15$), 1B ($N = 30$), and 1C ($N = 60$) were constructed with the 50 numbers of column 61 and the first 10 of column 62. The No. 2 samples were derived from columns 46 and 47, etc.

The breakdown of the 60 random normal numbers, e.g., from columns 61 and 62, was as follows, with the random numbers added to the predicted value for each of the nine test design points:

Test Design Point	Sample Numbers		
X_1, X_2	1A to 10A	1B to 10B	1C to 10C
(-1, -1)	1-7	1-14	1-28
(0, -1)	29	29,30	29-32
(1, -1)	33	33,34	33-36
(-1, 0)	37	37,38	37-40
(0, 0)	41	41,42	41-44
(1, 0)	45	45,46	45,48
(-1, 1)	49	49,50	49,52
(0, 1)	53	53,54	53,56
(1, 1)	57	57,58	57-60

As an example of the above, the detailed Example 1 was based on the 7A and 7B samples. The No. 7 samples were derived from columns 38 and 39 of the table of random normal numbers. Taking the (0, 0) point as an example, the predicted value from the basic equation is 14.61. Numbers 41 and 42 in column 38 are -1.75 and +1.23. Adding these to 14.61 gives the simulated test data values of 12.86 and 15.84, the (0, 0) values used in the example.

Sample Size Determining Experiments

Some of the factors which influence sample size were developed in the explanation of Example 1. It was seen that testing of only seven units at the temperature and vibration requirement point (-1, -1) gave an erroneous indication of the reliability of the design and that it was necessary to test an additional 10 units before the desired reliability was indicated. Similarly, testing only one unit at each of the other eight points of the test design gave an erroneous indication of the shape of the response surface, implying that the quadratic components were not significant.

It is desirable to keep the required sample size as small as possible, particularly when dealing with expensive units and tests. In order to gain insight into the minimum sample size which will give effective results, extensive empirical data is required, the purpose of this section. Thirty samples were established in accordance with the procedures given in the last section. The three sample sizes, 15, 30 and 60, were arbitrarily chosen as practical round numbers. The established test data for the samples is given in Appendix IIA. The remainder of Appendix II gives calculations, similar to those in Appendix I for Example 1, for the complete analysis of the test data for the 30 samples, ending with the reliability boundary equations in Appendices IID6, IIE6 and IIF5 from which the contours were plotted as shown in Figures 14, 15 and 16.

The inadequacy of a subsample of 7 units at (-1, -1) is clearly shown in Appendix IIB where 5 of the 10 "A" samples show inadequate reliability, i.e. $\bar{Y}$ - KS values below the 10 second requirement. This occurred with only one of the "B" samples (7B) with 14 tests at (-1, -1). As shown in the previous Example 1, based on the 7A and 7B samples, three more tests were required to establish the reliability. All of the "C" samples with 28 tests at (-1, -1) were adequate in this respect.

The inadequacy of the subsample of 7 at (-1, -1) is further illustrated by the tests of the assumption of normal distributions as shown on Figures 10, 11, 12 and 13 based on calculations given in Appendix IIC. Figure 10 shows in general a wide variation of the cumulative distribution points around the best, or least squares, straight lines, indicating little confidence in making the assumption of normality even when it is known, as in this case, that the points were derived from a known normal distribution. Figure 11, showing the plots for the "B" subsamples of 14, indicates that this number is about sufficient for making the required assumption of normality. Most of the solid points are close to the required straight lines. The up, down and up plot of sample 5B is characteristic of a double population distribution on either side of the mean value. This actually occurs in the column of random normal numbers from which the sample was derived. In practice, however, there is no known way in which a battery performance parameter can be distributed in this manner other than through chance sample variability. Therefore the plot of 5B could be considered to represent a straight line, thus satisfying the assumption of normality. The tendency most to be looked for in analyzing these plots is a pronounced curvature in one direction or the other, suggesting the probability of a skewed distribution. A slight tendency of this sort is noted in the plot of 10B, but not enough to counter the assumption of normality. The plots for the "C" samples in Figures 12 and 13 all tend to satisfy the assumption of normality of distribution.

"A" Sample Cumulative Frequency Distribution Plots

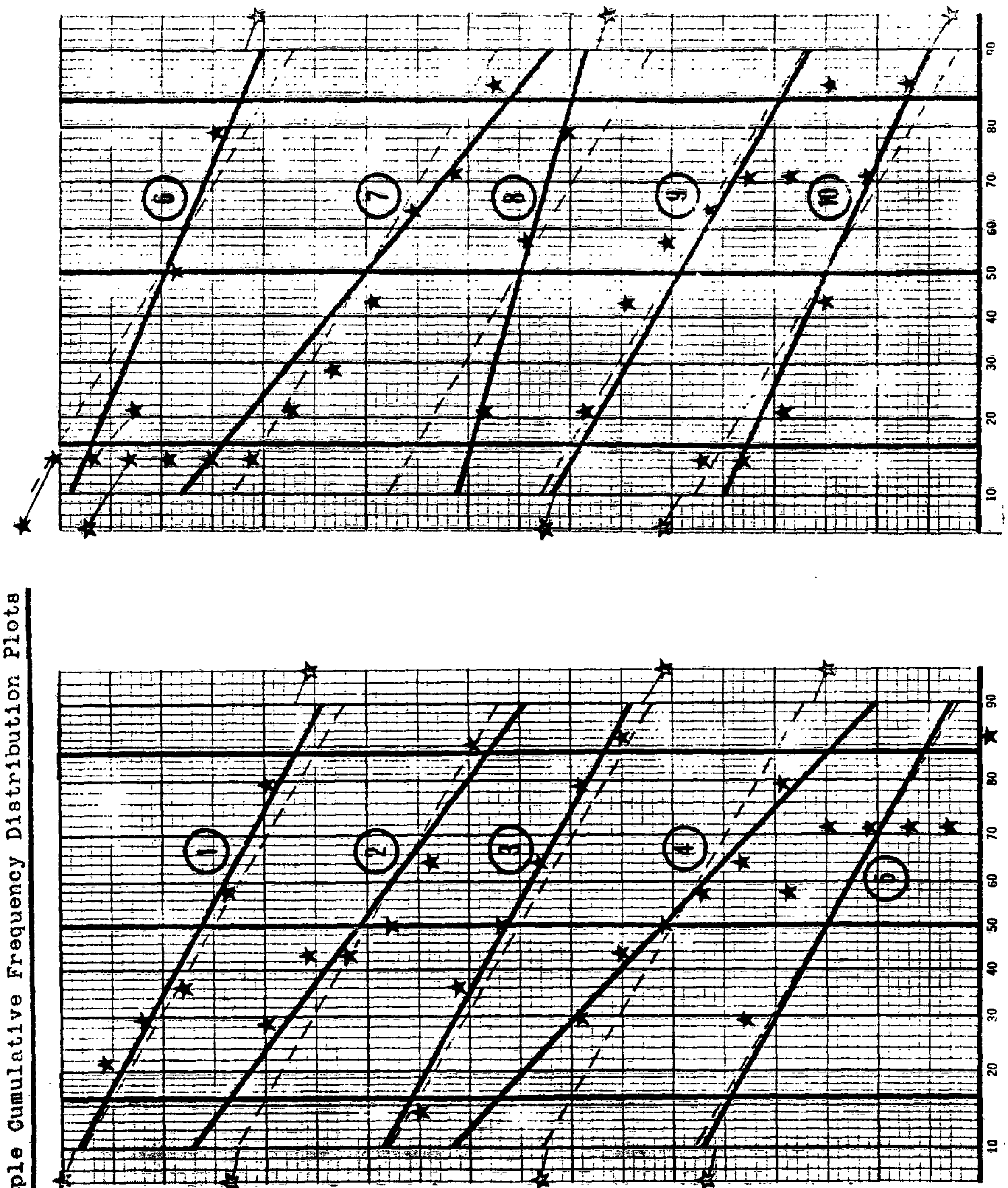


Figure 10

"B" Sample Cumulative Frequency Distribution Plots

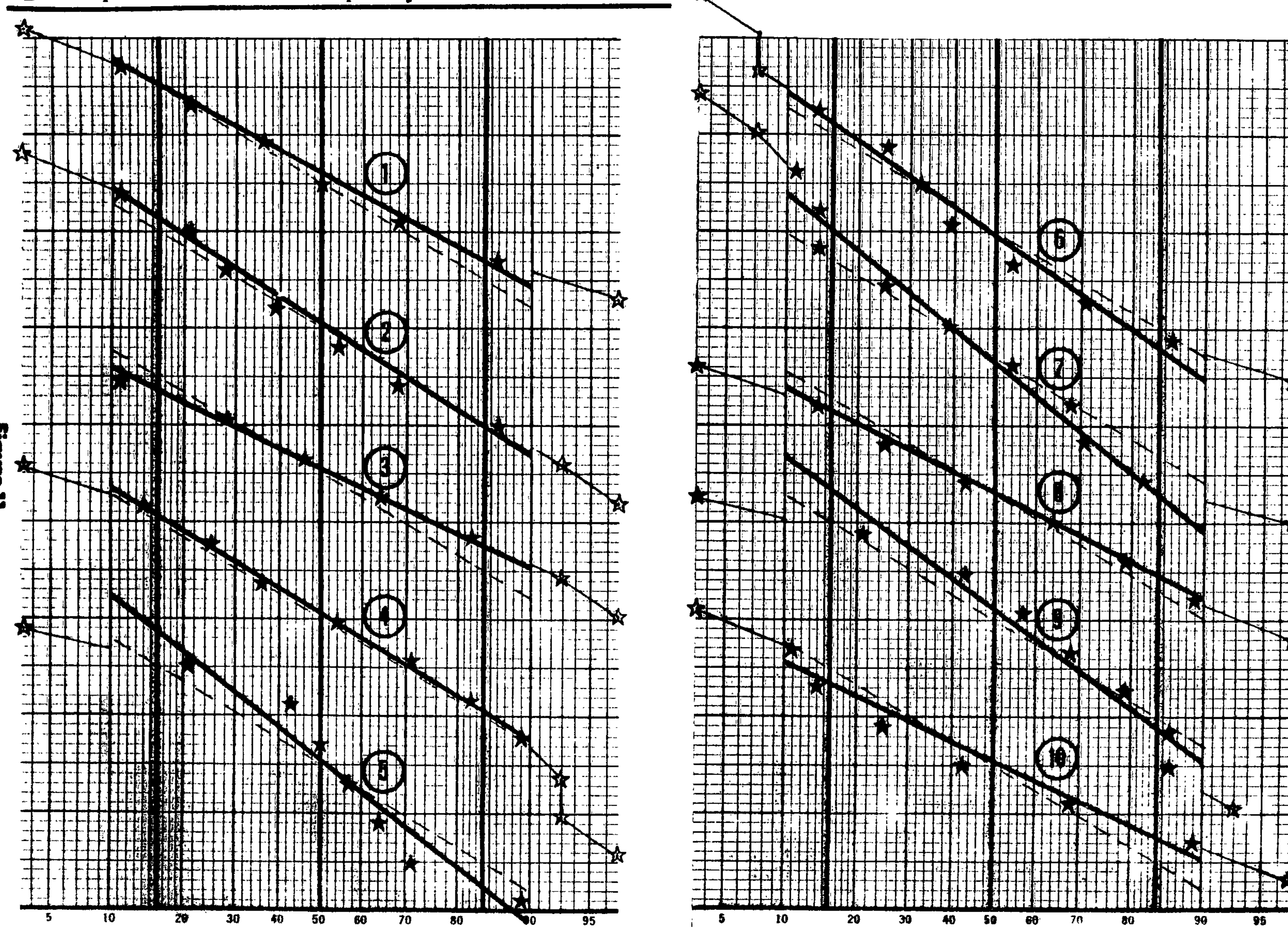


Figure 11

"C" Sample Cumulative Frequency Distribution Plots

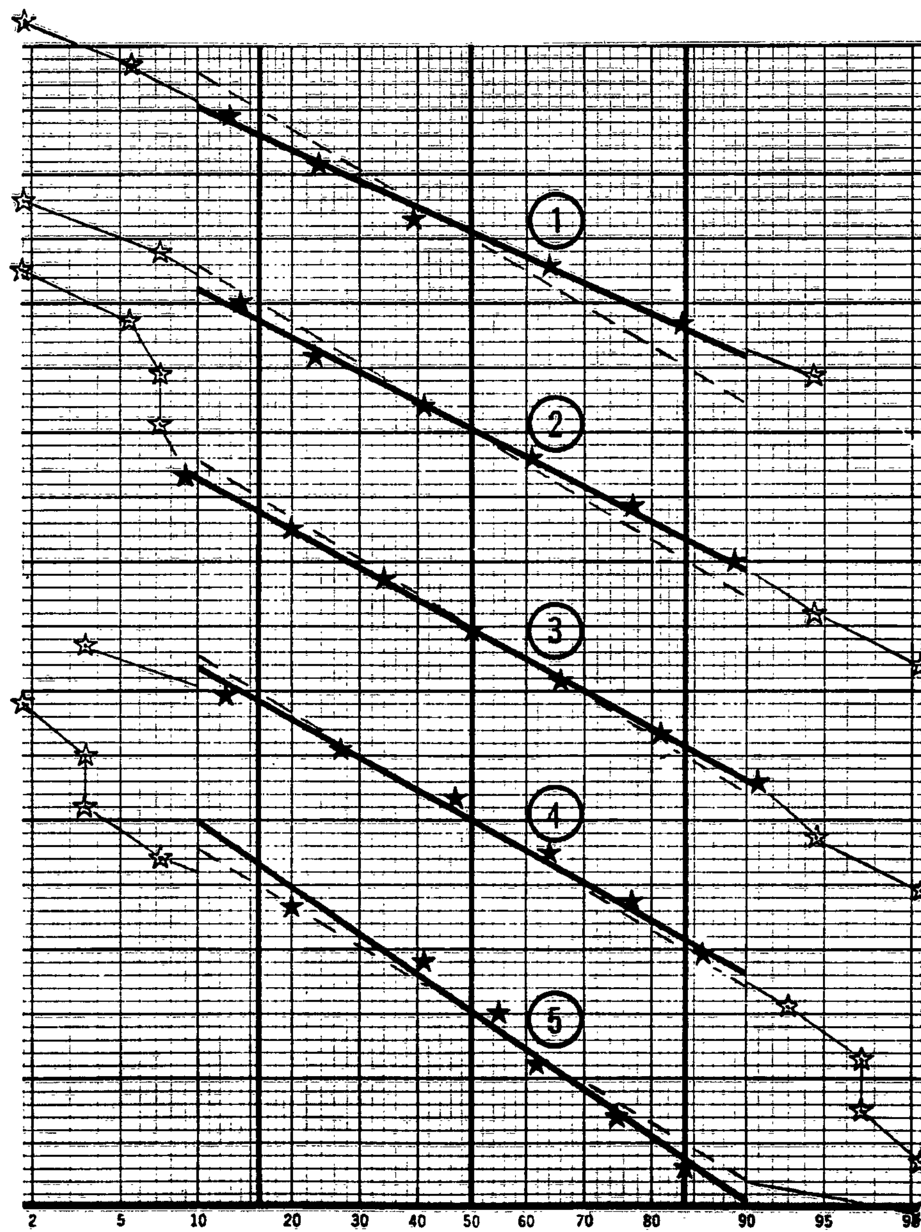


Figure 12

"C" Sample Cumulative Frequency Distribution Plots

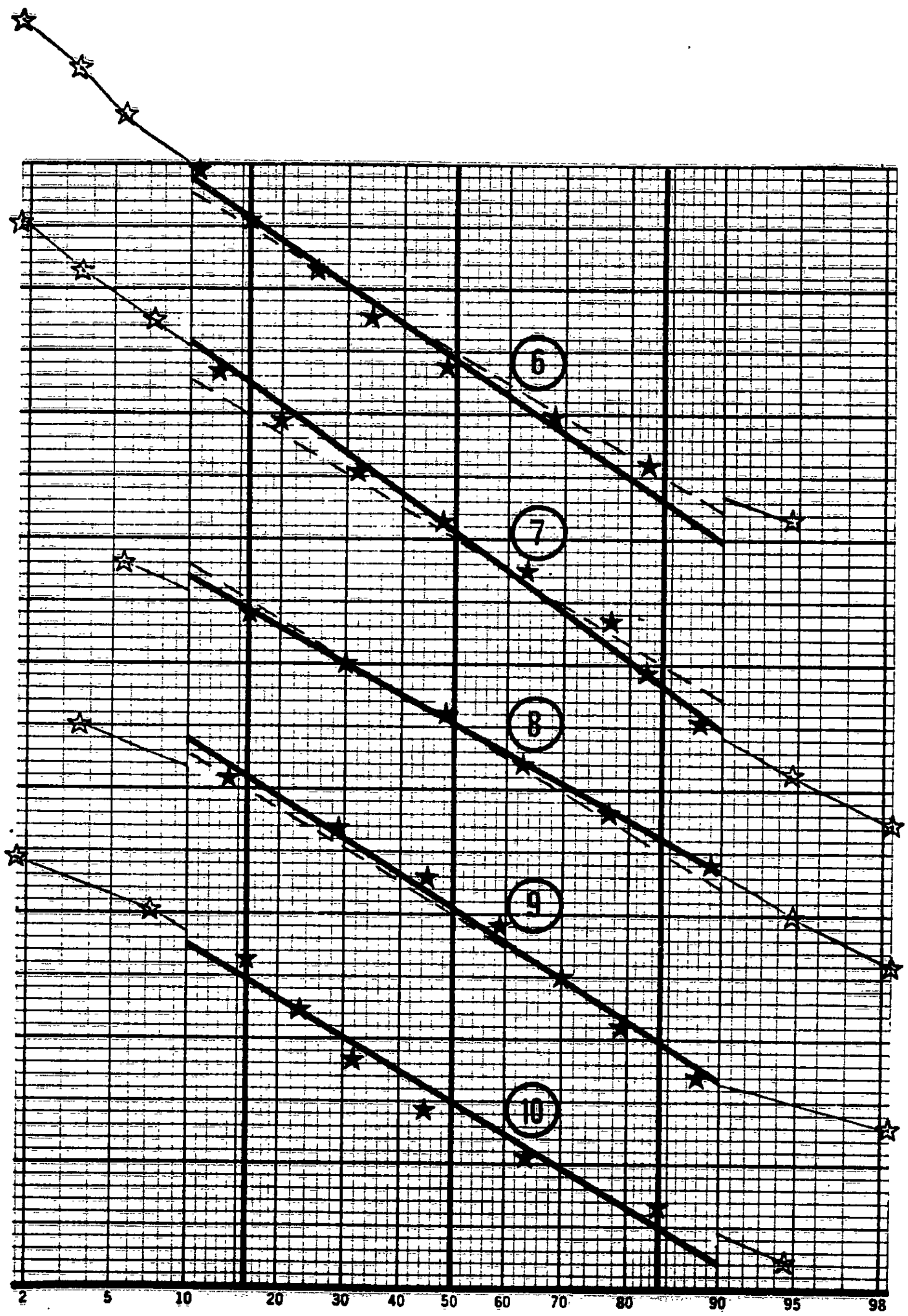


Figure 13

All of the cumulative distribution plots have dotted lines on them, all parallel to each other. These represent the straight line plots for an infinite sample size with a standard deviation of one and are shown to indicate how well the individual samples correspond to the true mean and standard deviation. It is seen that the larger the sample the closer the correspondence in general. The three vertical lines in each figure represent the mean, at $P = 50$, and the minus and plus one standard deviation values at $P = 16$ and $P = 84$.

Based on the foregoing, it may be stated that when dealing with samples drawn from a normally distributed population with relatively low variance, a sample of approximately 14 at (-1, -1) will in general satisfy the assumption of normality with a cumulative distribution plot which approximates a straight line. Slight departures from normality cannot reasonably be detected, but the probability is that relatively large degrees of skewness can be. Much more confidence in assuming a normal distribution can of course be had with a larger sample size if this is economically feasible.

In conducting an actual experiment, once it has been determined that it is not unreasonable to assume normality of distribution, at least in the region about (-1, -1), the remainder of the test region will be explored to estimate the nature of the response surface. The minimum subsample size for this is eight, one at each of the remaining test design points. Even for a sample drawn from a population with relatively low variance, Example 1 showed that this minimum subsample size is inadequate. This is more positively demonstrated by Figure 14, the "A" sample reliability boundaries drawn from the calculations given in Appendix IID. Nine of the 10 boundaries fall above (-1, -1) but the wide variation of them around curve B indicates a high probability of one of these small samples indicating that the battery design is not reliable when it actually is. Sample 7A is in this category. The mean failure contour for the sample, as shown by the dots, gives a good estimate of the true mean failure contour A. In fact each of the 10 samples do, as shown by the cluster of points around curve A at $X_2 = 0$. The wide variation in the reliability boundaries is due to the variation which the samples give in the estimate of the standard error. There is also a wide variation in the estimates of the coefficients of the response surface equations (Appendix IID1). This accounts for the considerable variation in the shapes of the contours. Another indication of the inadequacy of the sample size is given in the Analysis of Variance in Appendix IID2. Only four of the samples show the quadratic terms to be significant. For the others linear response surfaces, or planes, are erroneously indicated with straight line contours.

Replication, repeating the experiment, is found to eliminate all of these problems as shown with the reliability boundaries for the "B" samples as given in Figure 15, based on the calculations in Appendix IIE. It is seen that each of the 10 samples now gives a fairly good estimate of the true contour, represented by curve B. The Analysis of Variance given in Appendix IIE2 shows that each sample correctly indicates that the quadratic components of the mathematical model are significant. Having two results at each test design point also affords a measure of the fit of the mathematical model to the data. In every case the lack of fit term is found to be insignificant. The assumption of uniformity of variance throughout the entire test region is satisfactorily met for each sample as shown in Appendix IIE5. Again, uniform variance is inferred by the inability to prove the variance at (-1,-1)

1A to 10A Sample Reliability Boundaries, 1 Test per Design Point

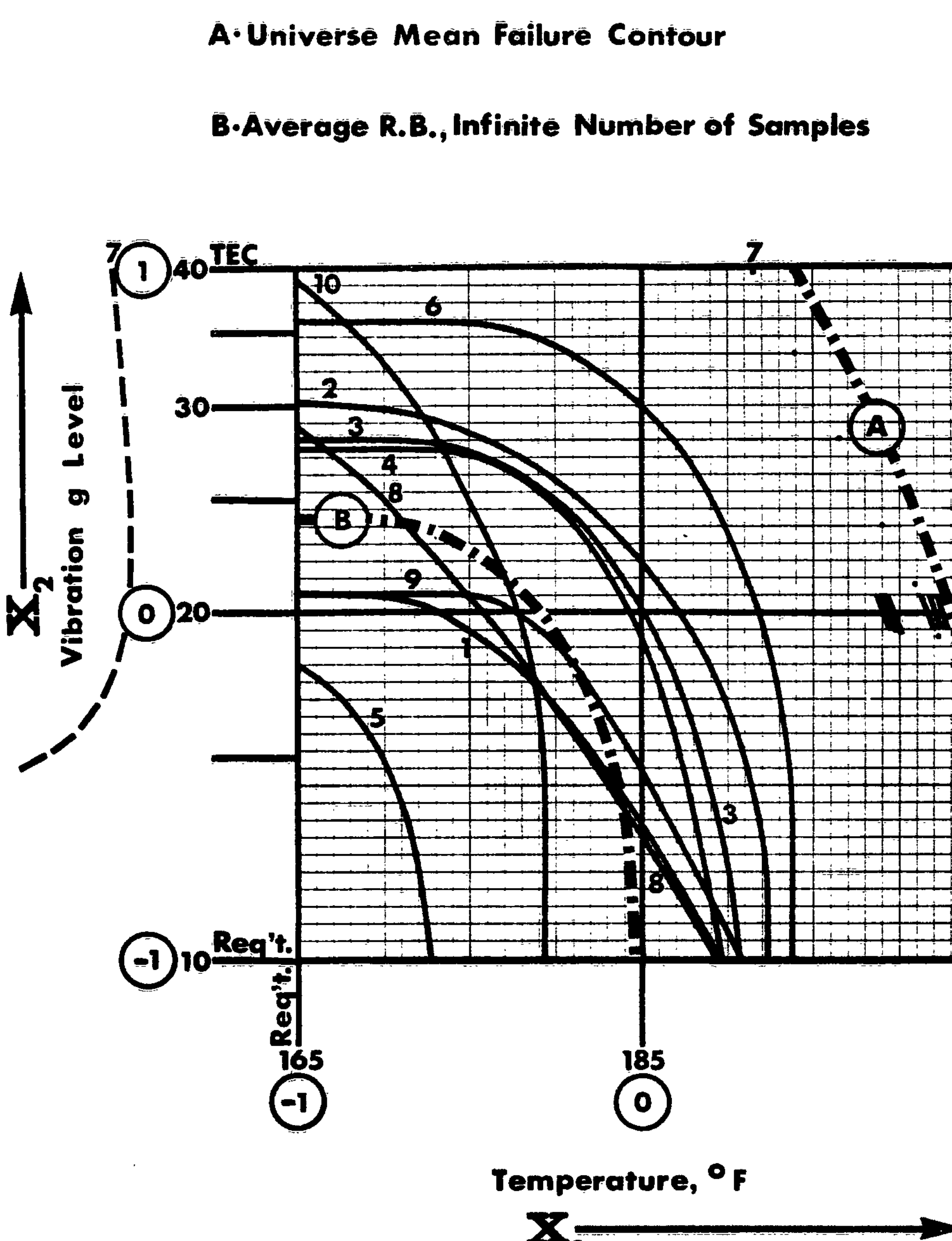


Figure 14

1B to 10B Sample Reliability Boundaries, 2 Tests per Design Point

A· Universe Mean Failure Contour

B· Average R.B., Infinite Number of Samples

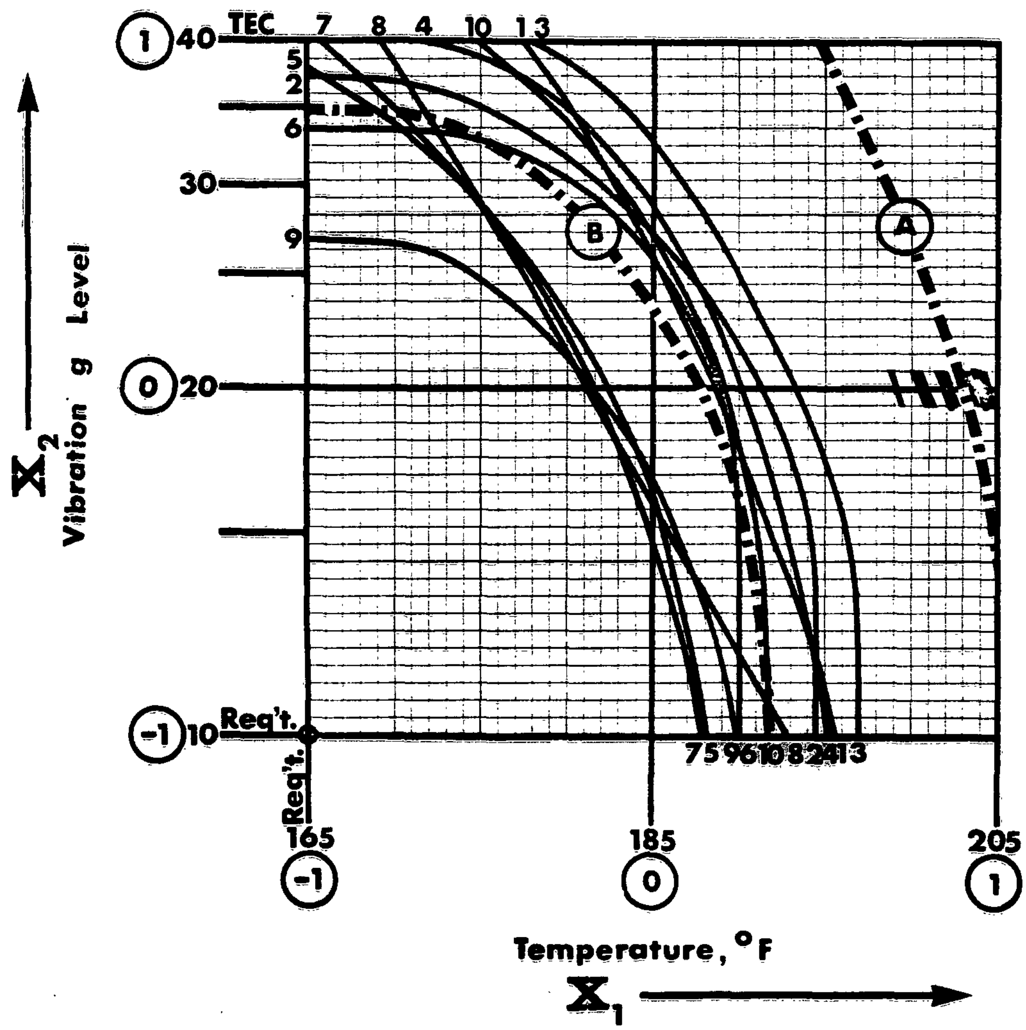


Figure 15

to be significantly different from the variance throughout the rest of the test region. This makes it possible to compute a uniform pooled standard error based on all of the test data and from this to calculate the equations for the reliability boundaries.

The reliability boundaries for the "C" samples, with four tests at each of the eight test points, as given in Figure 16, naturally show a still closer estimate of the true curve B. Whether they give that much better an estimate than the "B" samples do to justify the added costs is a debatable point. In the opinion of the author, they do not. However, for a battery design with much greater variability than the hypothetical design, or for samples which give reliability boundaries on or slightly below (-1, -1), it would undoubtedly be necessary to replicate beyond two tests at each point.

An analysis of variance is not included in Appendix IIF for the "C" samples since the analysis for the "B" samples had shown these to be satisfactory in regard to significance of the quadratic terms and proper fit of the mathematical model.

The effects of replication are shown quite clearly on Figure 17. The "A" sample reliability boundary 7.1 gives a poor estimate of the true curve 1A (curve B of Figure 14). The "B" sample boundary gives a much better estimate of the true curve 2A and this is even more pronounced with the "C" sample curves 7.4 and 4A. Additional curves are shown for a tremendous sample consisting of 40 tests at each of the eight design points. Contour 40 corresponds very closely, as one would expect, with 40A representing the average reliability boundary of an infinite number of samples with 40 at each point. Curve 40 M.F.C., the mean failure contour, is found to correspond perfectly with the total population or universe mean failure contour. The curves 1A, 2A, 4A, and 40A are seen to approach a limiting contour which is called the U.R.B., or universe reliability boundary. This curve is derived from an equation based on a K factor of 3.09 for a sample size of infinity. It may be interpreted by saying that in testing an infinite sample size, one out of a thousand units would fall below the U.R.B. Fifty per cent would, of course, fall below the universe mean failure contour.

1C to 10C Sample Reliability Boundaries, 4 Tests per Design Point

A- Universe Mean Failure Contour

B- Average R.B., Infinite Number of Samples

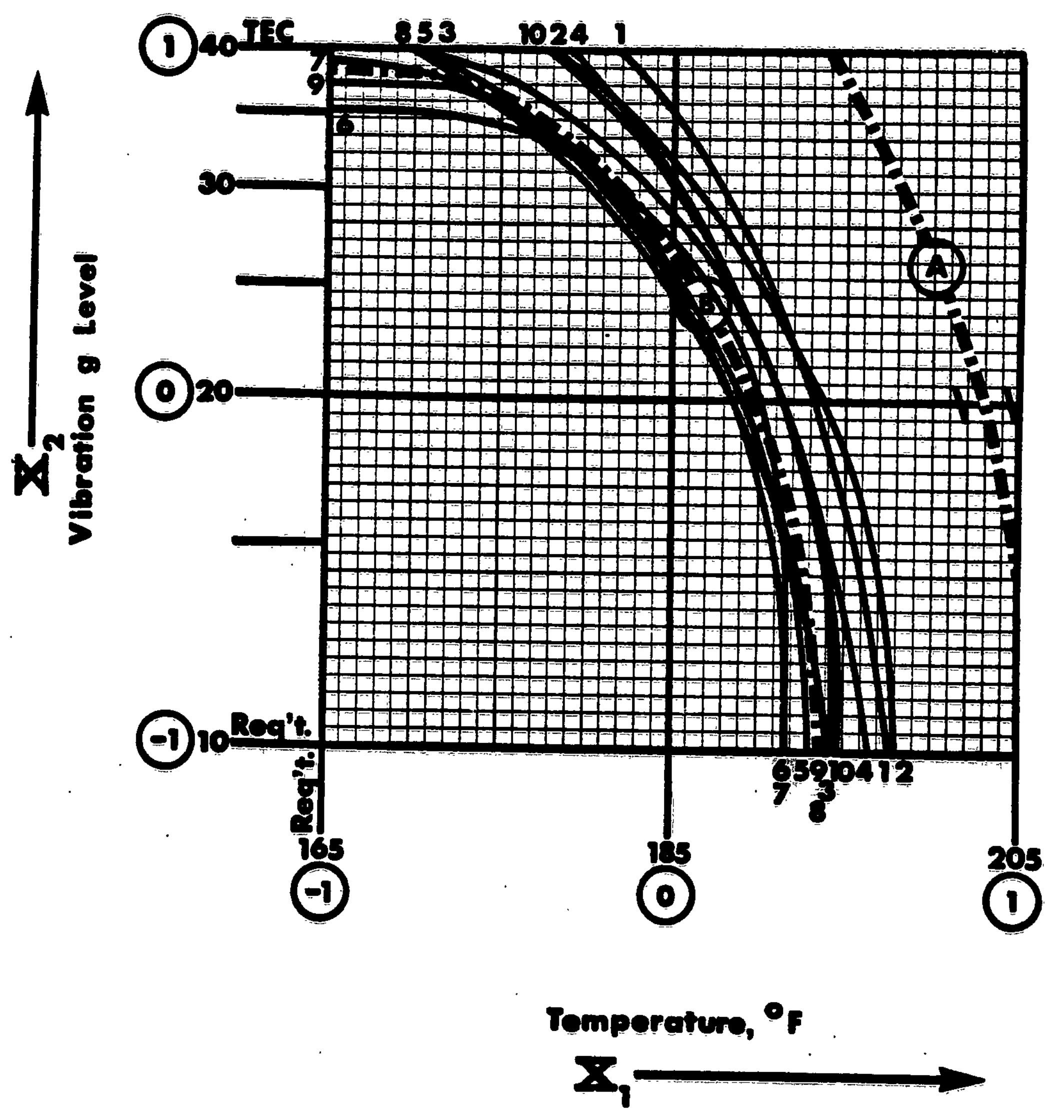


Figure 16

Effect of Replication on R.B. Estimates from No.7 Samples

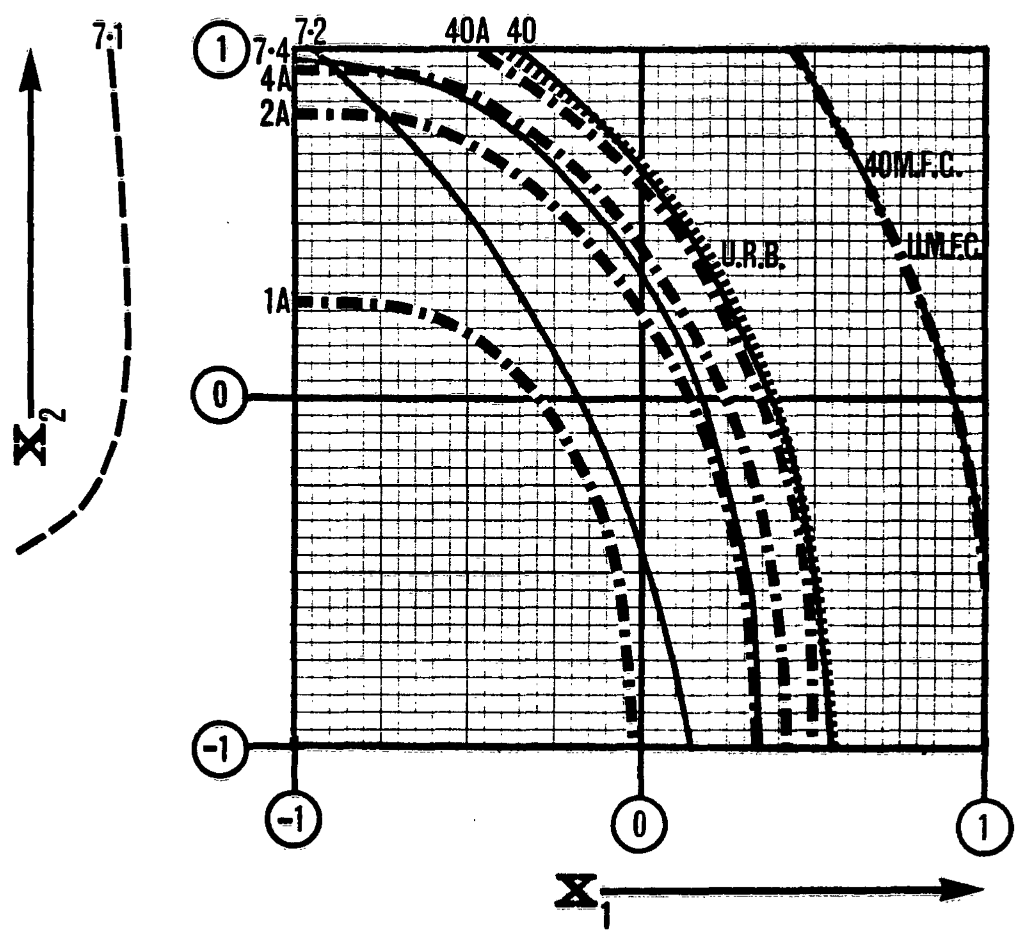


Figure 17

CONCLUSIONS

The theory of the RSD method has been covered, the mathematics involved have been explained in detail, and an analysis of the effects of sample size has been performed in order to provide a guide in determining sample sizes for an actual reliability program for batteries or other one-shot items. It has been demonstrated that the method will provide a great deal of valuable reliability prediction data from the testing of a relatively small sample.

As posed at the end of the first section of the discussion, a basic question is, what assurance is there that the final reliability predictions are correct and usable? This is particularly relevant since the sample test data was not only used for making the predictions, but for testing the assumptions of normality of distribution and uniformity of variance upon which basis the predictions were made. The sample size determining section showed that the sample size has great bearing on the assurance with which the final conclusions may be accepted. The smallest sample size of 15 showed a relatively high probability of erroneously concluding that the battery design was unreliable. The wide scatter of the sample reliability boundaries also showed that little faith could be placed in reliability estimates out along the X axes. The predictions for $\bar{N} = 60$ showed such excellent conformance with the true picture that the reliability predictions may be made with great assurance in their accuracy. However, at least for most missile battery programs, a sample size of 60 would be considered excessive from an economic standpoint. Forty-five would probably be as well. Thirty would be considered a reasonable size in many cases.

The results for the "B" samples of size 30 showed in general a good approximation of the characteristics of the battery population. The test of the suitablility of assuming normal distribution appeared satisfactory, as well as the test for significance of quadratic components, the test of the assumption of uniform variance and the actual reliability boundaries themselves. These factors indicate that the sample size of 30 should be adequate in most cases and that the final reliability predictions can be made with considerable assurance. In conducting an actual program, as previously pointed out, there may be instances where more than 14 tests will be required at the X_1, X_2 requirement point and where it may be necessary to replicate beyond two tests per point throughout the rest of the test design. In establishing the total lot size for the program, a reserve should be provided for these contingencies.

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APPENDIX I

Experiment 1

A. Analysis of responses at $X_1 = 165^\circ$ F, $X_2 = 10g$

$$\Sigma Y = 111.16$$

$$\bar{Y} = 15.88$$

$$\Sigma Y^2 = 1,778.5098$$

$$\frac{(\Sigma Y)^2}{7} = 1,765.2208$$

$$S = \sqrt{\frac{\Sigma Y^2 - \frac{(\Sigma Y)^2}{N}}{N-1}} = \sqrt{\frac{1,778.5098 - 1,765.2208}{6}} = \sqrt{2.21483} = 1.488$$

B. Cumulative distribution points of responses at $X_1 = 165^\circ$F, $X_2 = 10g$

Select suitable intervals for Y such that there are from 10 to 20 intervals. Then set up a table for the midpoints of the selected Y intervals, tally the Y responses and set up three additional columns for frequency, cumulative frequency, and cumulative frequency expressed as percentages of $2 \times N = 34$. The cumulative frequency column is derived from the frequency column by a counterclockwise rotating addition system; e.g.,

$$\begin{array}{ccc} 0+0 & 1+1 & 0+2 \\ + & + & + \\ 1=1, & 0=2, & 2=4, \end{array} \quad \text{etc.}$$

The cumulative frequency values thus obtained are divided by the double total to yield the cumulative frequency percentages (expressed to tenths of a per cent below 10% and above 90%).

Y	Tally	Frequency	Cumulative Frequency	Cumulative Frequency (% of 2N = 34)
19.0		0	0	--
18.6	1	1	1	2.9
18.2		0	2	5.9
17.8	11	2	4	12
17.4	1	1	7	21
17.0		0	8	24
16.6	111	3	11	32
16.2	11	2	16	47
15.8	111	3	21	62
15.4	1	1	25	74
15.0		0	26	76
14.6	11	2	28	82
14.2	11	2	32	94.1
13.8		0	34	--

C. <u>Determination of coefficients of a fitted model of the form :</u>

$$Y = b_o + b_1X_1 + b_2X_2 + b_{11}X_1^2 + b_{22}X_2^2 + b_{12}X_1X_2$$

X_1	X_2	Y	X_1Y	X_2Y	X_1^2Y	X_2^2Y	X_1X_2Y
-1	-1	16.12	-16.12	-16.12	16.12	16.12	16.12
0	-1	16.07	0	-16.07	0	16.07	0
1	-1	11.27	11.27	-11.27	11.27	11.27	-11.27
-1	0	16.99	-16.99	0	16.99	0	0
0	0	12.86	0	0	0	0	0
1	0	9.19	9.19	0	9.19	0	0
-1	1	15.78	-15.78	15.78	15.78	15.78	-15.78
0	1	11.22	0	11.22	0	11.22	0
1	1	6.43	6.43	6.43	6.43	6.43	6.43
SUMS		115.93	-22.00	-10.03	75.78	76.89	-4.50

$$b_1 = \frac{\Sigma X_1 Y}{6} = \frac{-22.00}{6} = -3.67$$

$$b_2 = \frac{\Sigma X_2 Y}{6} = \frac{-10.03}{6} = -1.67$$

$$b_{12} = \frac{\Sigma X_1 X_2 Y}{4} = \frac{-4.50}{4} = -1.13$$

$$b_{11} = \frac{\Sigma X_1^2 Y - 2/3\Sigma Y}{2} = \frac{75.78-77.29}{2} = \frac{-1.51}{2} = -0.76$$

$$b_{22} = \frac{\Sigma X_2^2 Y - 2/3\Sigma Y}{2} = \frac{76.89-77.29}{2} = \frac{-0.40}{2} = -0.20$$

$$b_0 = \frac{\Sigma Y}{9} - 2/3 b_{11} - 2/3 b_{22} = 12.881 + 0.507 + 0.133 = 13.52$$

Therefore the fitted equation is:

$$Y = 13.52 - 3.67X_1 - 1.67X_2 - 0.76X_1^2 - 0.20X_2^2 - 1.13X_1X_2$$

A check is made for arithmetical errors. The sum of the coefficients is 6.09. This is checked against the addition of the Y response values, ordered from Y_1 to Y_9, in accordance with the following formula:

$$\sum \text{Coefficients} = \frac{5Y_1 - 4(Y_2 + Y_4 + Y_5) - Y_3 + 8(Y_6 + Y_8) - Y_7 + 29Y_9}{36}$$

$$= \frac{5 \times 16.12 - 4(16.07 + 16.99 + 12.86) - 11.27 + 8(9.19 + 11.22) - 15.78 + 29 \times 6.43}{36}$$

$$= \frac{80.60 - 183.68 - 11.27 + 163.28 - 15.78 + 186.47}{36} = \frac{219.62}{36} = 6.10$$

Allowing for a difference due to rounding off errors, this check indicates that no arithmetical errors have been made in establishing the coefficients.

D. Analysis of variance

Source of Variance	Sums of Squares	Degrees of Freedom (d.f.)	Mean Square
ΣY^2	1,599.8497	9	
due to b_0, $\frac{(\Sigma Y)^2}{9}$	1,493.3072	1	
due to b_1, $\frac{(\Sigma X_1 Y)^2}{6}$	80.6667	1	
due to b_2, $\frac{(\Sigma X_2 Y)^2}{6}$	16.7668	1	
quadratic components			
due to b_{11}, $\frac{(\Sigma X_1^2 Y - 2/3\Sigma Y)^2}{2}$	1.1401	1	
due to b_{22}, $\frac{(\Sigma X_2^2 Y - 2/3\Sigma Y)^2}{2}$	0.0800 6.2826	1 3	2.0942
due to b_{12}, $\frac{(\Sigma X_1 X_2 Y)^2}{4}$	5.0625	1	*
Residual	2.8264	3	0.9421

$$^{*}F_{3,3} = \frac{2.0942}{0.9421} = 2.22$$

In order for the quadratic components to be significant at a 95% significance level, the $F_{3,3}$ ratio must be at least 9.28. The ratio of 2.22 shows the quadratic components to be not significant. Another way of stating this is that the quadratic components have not been proved to be significantly different from zero.

E. Determination of coefficients of a fitted model of the form:

$$Y = b_0 + b_1 X + b_2 X_2 + b_{11} X_1^2 + b_{22} X_2^2 + b_{12} X_1 X_2$$

X_1	X_2	Y	X_1Y	X_2Y	X_1^2Y	X_2^2Y	X_1X_2Y
-1	-1	16.12	-16.12	-16.12	16.12	16.12	16.12
		16.12	-16.12	-16.12	16.12	16.12	16.12
0	-1	16.07	0	-16.07	0	16.07	0
		16.55	0	-16.55	0	16.55	0
1	-1	11.27	11.27	-11.27	11.27	11.27	-11.27
		9.87	9.87	-9.87	9.87	9.87	-9.87
-1	0	16.99	-16.99	0	16.99	0	0
		15.79	-15.79	0	15.79	0	0
0	0	12.86	0	0	0	0	0
		15.84	0	0	0	0	0
1	0	9.19	9.19	0	9.19	0	0
		8.72	8.72	0	8.72	0	0
-1	1	15.78	-15.78	15.78	15.78	15.78	-15.78
		15.09	-15.09	15.09	15.09	15.09	-15.09
0	1	11.22	0	11.22	0	11.22	0
		11.05	0	11.05	0	11.05	0
1	1	6.43	6.43	6.43	6.43	6.43	6.43
		6.12	6.12	6.12	6.12	6.12	6.12
		231.08	-44.29	-20.31	147.49	151.69	-7.22

$$b_1 = \frac{\Sigma X_1 Y}{12} = -\frac{44.29}{12} = -3.69$$

$$b_2 = \frac{\Sigma X_2 Y}{12} = -\frac{20.31}{12} = -1.69$$

$$b_{12} = \frac{\Sigma X_1 X_2 Y}{8} = -\frac{7.22}{8} = -0.90$$

$$b_{11} = \frac{\Sigma X_1^2 Y - 2/3\Sigma Y}{4} = \frac{147.49-154.05}{4} = \frac{-6.56}{4} = -1.64$$

$$b_{22} = \frac{\Sigma X_2^2 Y - 2/3\Sigma Y}{4} = \frac{151.69-154.05}{4} = -\frac{2.36}{4} = -0.59$$

$$b_0 = \frac{\Sigma Y}{18} - 2/3 b_{11} - 2/3 b_{22} = 12.838 + 1.093 + 0.393 = 14.32$$

Therefore the fitted equation is:

$$Y = 14.32 - 3.69X_1 - 1.69X_2 - 1.64X_1^2 - 0.59X_2^2 - 0.90X_1X_2$$

Check: ΣCoefficients = 5.81

$$\text{Coefficients} = \frac{5Y_1 - 4(Y_2+Y_4+Y_5) - Y_3 + 8(Y_6+Y_8) - X_7 + 29Y_9}{72 \text{ (for } N = 18)}$$

$$= \frac{5(32.24) - 4(94.10) - 21.14 + 8(40.18) - 30.87 + 29(12.55)}{72}$$

$$= \frac{161.20 - 376.40 - 21.14 + 321.44 - 30.87 + 363.95}{72}$$

$$= \frac{418.18}{72} = 5.81$$

F. **Analysis of variance**

Source of Variance	Sums of Squares	Degrees of Freedom (d.f.)	Mean Square
ΣY^2	3,194.5566	18	
due to b_0, $\frac{(\Sigma Y)^2}{18}$ *	2,966.5537	1	
due to b_1, $\frac{(\Sigma X_1 Y)^2}{12}$	163.4670	1	
due to b_2, $\frac{(\Sigma X_2 Y)^2}{12}$	34.3747	1	
quadratic components			
due to b_{11}, $\frac{(\Sigma X_1^2 Y - 2/3\Sigma Y)^2}{4}$	10.7584	1	
due to b_{22}, $\frac{(\Sigma X_2^2 Y - 2/3\Sigma Y)^2}{4}$	1.3924 18.6669	1 3	6.2223
due to b_{12}, $\frac{(\Sigma X_1 X_2 Y)^2}{8}$	6.5161	1	**
Residual	11.4943	12	0.9579
Lack of Fit	4.8279	4	1.2070
Experimental Error	6.6664	8	0.8333

*Denominators doubled, in comparison to Appendix 1D, since test results are doubled.

$F_{4,8} = 1.45$

** $F_{3,12} = 6.50$

$$\text{Experimental Error Sum of Squares} = \frac{\sum(\text{differences})^2}{2}$$

$$= \frac{(0.48)^2 + (1.40)^2 + (1.20)^2 + (2.98)^2 + (0.47)^2 + (0.69)^2 + (0.17)^2 + (0.41)^2}{2}$$

$$= \frac{13.3328}{2} = 6.6664$$

The critical $F_{3,12}$ ratio at a 95% level of significance is 3.49. This is exceeded by the 6.50 ratio. Therefore, the quadratic components are significant. The lack of fit ratio, 1.45, is less than $F_{4,8} = 3.84$, therefore the lack of fit term is not significantly different from experimental error and, therefore, is non-significant.

G. Determination of standard error, all test points except (-1, -1)

Test Point X_1	X_2	Y Predicted Value (From equation)	y Test Results	Differences (Y-y) (Signs not necessary)
0	-1	15.43	16.07	0.64
			16.55	1.12
1	-1	11.00	11.27	0.27
			9.87	1.13
-1	0	16.38	16.99	0.61
			15.79	0.59
0	0	14.33	12.86	1.47
			15.84	1.51
1	0	8.98	9.19	0.21
			8.72	0.26
-1	1	15.00	15.78	0.78
			15.09	0.09
0	1	12.03	11.22	0.81
			11.05	0.98
1	1	5.76	6.43	0.67
			6.12	0.36

$$\Sigma(Y-y)^2 = 11.0982$$

$$\text{Variance} = \frac{\Sigma(Y-y)^2}{11(\text{d.f.})}$$

$$= 1.00893$$

$$S = \sqrt{\frac{\Sigma(Y-y)^2}{11}} = 1.004$$

Note: 5 d.f. are expended for estimating the coefficients; therefore, the denominator is N-5 = 16-5 = 11 . The sixth coefficient is based on a d.f. from the tests at (-1, -1).

H. Standard error from pooling of estimates of variance

$$S_p^{\,2} \text{ (pooled variance)} = \frac{11(\text{d.f.}) \times 1.00893 + 16(\text{d.f.}) \times 1.72456}{27 \text{ d.f.}}$$

$$= \frac{11.09823 + 27.59296}{27} = \frac{38.69119}{27} = 1.43301$$

$$S_p \text{ (pooled standard error)} = \sqrt{S_p^{\,2}} = 1.197$$

I. Determination of Mean Failure Contour Points

1. **Mean failure contour equation:**

$$3.70X_1 + 1.70X_2 + 1.65X_1^2 + 0.60X_2^2 + 0.92X_1X_2 = 4.33$$

2. Value of X_1 at $X_2 = 0$; $3.70X_1 + 1.65X_1^2 = 4.33$

   estimate X_1 at 0.80 ; $2.96 + 1.06 = 4.02$

   estimate X_1 at 0.85 ; $3.15 + 1.19 = 4.34$

   estimate X_1 at 0.84 ; $3.11 + 1.16 = 4.27$, $\therefore$ root is $0.85 = X_1$

3. Value of X_2 at $X_1 = 0$; $1.70X_2 + 0.60X_2^2 = 4.33$

   estimate X_2 at 2.5 (beyond test region, maximum $X_2 = 1.0$)

4. Value of X_1 at $X_2 = 1$; $3.70X_1 + 1.70 + 1.65X_1^2 + 0.60 + 0.92X_1 = 4.33$

$$4.62X_1 + 1.65X_1^2 = 4.33 - 1.70 - 0.60 = 2.03$$

   estimate X_1 at 0.40 ; $1.85 + 0.26 = 2.11$

   estimate X_1 at 0.38 ; $1.76 + 0.24 = 2.00$

   estimate X_1 at 0.39 ; $1.80 + 0.25 = 2.05$, $\therefore$ root is $0.39 = X_1$

5. Value of X_1 at $X_2 = -1$; $2.78X_1 + 1.65X_1^2 = 4.33 + 1.70 - 0.60 = 5.43$

   estimate X_1 at 1.50 ; $4.17 + 3.71 = 7.88$

   estimate X_1 at 1.20 ; $3.34 + 2.38 = 5.72$

   estimate X_1 at 1.10 ; $3.06 + 2.00 = 5.06$

   estimate X_1 at 1.15 ; $3.20 + 2.18 = 5.38$

   estimate X_1 at 1.16 ; $3.22 + 2.22 = 5.44$, $\therefore$ root is $1.16 = X_1$

Note: Extraction of the roots by the above trial-and-error method is very fast with a desk calculator, much faster than using the standard formula

$$X = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

J. **Determination of reliability boundary ($\bar{Y}$ - KS) contour points**

1. **Reliability boundary equation:**

$$3.70X_1 + 1.70X_2 + 1.65X_1^2 + 0.60X_2^2 + 0.92X_1X_2 = -0.57$$

2. **Value of X_1 at $X_2 = 0$**; $3.70X_1 + 1.65X_1^2 = -0.57$

estimate X_1 at -0.20 ; $-0.74 + 0.07 = -0.67$

estimate X_1 at -0.18 ; $-0.67 + 0.05 = -0.62$

estimate X_1 at -0.16 ; $-0.59 + 0.04 = -0.55$

estimate X_1 at -0.17 ; $-0.63 + 0.05 = -0.58$, $\therefore$ root is $-0.17 = X_1$

3. **Value of X_2 at $X_1 = 0$**; $1.70X_2 + 0.60X_2^2 = -0.57$

estimate X_2 at -0.40 ; $-0.68 + 0.10 = -0.58$

estimate X_2 at -0.39 ; $-0.66 + 0.09 = -0.57$, $\therefore$ root is $-0.39 = X_2$

4. **Value of X_1 at $X_2 = -1$**; $2.78X_1 + 1.65X_1^2 = -0.57 + 1.70 - 0.60 = 0.53$

estimate X_1 at 0.20 ; $0.56 + 0.07 = 0.63$

estimate X_1 at 0.18 ; $0.50 + 0.05 = 0.55$

estimate X_1 at 0.17 ; $0.47 + 0.05 = 0.52$, $\therefore$ root is $0.17 = X_1$

5. **Value of X_1 at $X_2 = 0.5$**; $4.16X_1 + 1.65X_1^2 = -0.57 - 0.85 - 0.15 = -1.57$

estimate X_1 at -0.50 ; $-2.08 + 0.41 = -1.67$

estimate X_1 at -0.40 ; $-1.66 + 0.26 = -1.50$

estimate X_1 at -0.45 ; $-1.87 + 0.33 = -1.54$

estimate X_1 at -0.47 ; $-1.96 + 0.36 = -1.60$, $\therefore$ root is $-0.46 = X_1$

6. **Value of X_1 at $X_2 = 1.0$**; $4.62X_1 + 1.65X_2 = -0.57 - 1.70 - 0.60 = -2.87$

estimate X_1 at -1.0 ; $-4.62 + 1.65 = -2.97$

estimate X_1 at -0.96 ; $-4.44 + 1.52 = -2.92$

estimate X_1 at -0.94 ; $-4.34 + 1.46 = -2.88$

estimate X_1 at -0.93 ; $-4.30 + 1.43 = -2.87$, $\therefore$ root is $-0.93 = X_1$

APPENDIX II

A. Sample Test Data

1. "A" Samples

Test Design Point	SAMPLE 1A	2A	3A	4A	5A	6A	7A	8A	9A	10A
(-1,-1)	16.09 15.72 16.98 16.54 18.35 17.86 15.62	15.21 16.40 16.54 14.55 14.99 14.78 17.06	15.08 16.57 14.17 15.77 16.00 16.41 15.34	16.36 14.95 16.94 17.07 14.74 15.38 17.39	14.48 16.45 17.28 16.87 16.99 16.76 14.76	15.86 15.87 15.32 17.45 14.95 15.85 16.19	15.74 15.63 14.41 16.61 15.47 14.51 18.79	15.98 15.50 16.27 15.96 16.23 14.83 16.38	15.55 16.24 16.22 16.85 16.56 14.38 14.37	15.77 16.06 15.04 17.20 15.54 15.86 14.79
(0,-1)	16.94	16.44	16.36	15.18	14.16	16.53	16.07	14.40	16.03	14.11
(1,-1)	9.93	11.30	10.49	11.05	11.62	9.21	11.27	11.72	9.65	9.43
(-1,0)	15.77	15.80	16.83	14.93	16.38	16.32	16.99	15.88	14.20	14.62
(0,0)	13.40	15.05	14.31	15.60	14.50	15.00	12.86	13.61	13.51	14.89
(1, 0)	7.71	10.61	8.04	9.36	8.13	9.71	9.19	7.00	8.03	8.93
(-1, 1)	13.77	13.88	12.37	13.77	13.40	13.47	15.78	13.39	12.75	15.86
(0, 1)	11.92	10.85	12.60	13.39	11.18	12.36	11.22	12.11	12.43	11.06
(1, 1)	6.61	4.74	5.61	7.65	6.66	5.58	6.43	5.97	4.14	5.37

2. **"B" Samples**

SAMPLE

Test Design Point	1B	2B	3B	4B	5B	6B	7B	8B	9B	10B
(-1,-1)	16.09 15.72 16.98 16.54 18.35 17.86 15.62 16.05 17.23 17.34 16.05 15.45 15.92 16.92	15.21 16.40 16.54 14.55 14.99 14.78 17.06 17.36 15.68 15.18 15.84 14.88 15.73 13.90	15.08 16.57 14.17 15.77 16.00 16.41 15.34 15.61 15.57 16.16 15.80 15.17 15.02 16.46	16.36 14.95 16.94 17.07 14.74 15.38 17.39 15.61 16.05 13.53 15.89 15.91 15.43 16.42	14.48 16.45 17.28 16.87 16.99 16.76 14.76 15.11 14.66 17.18 15.74 14.84 17.16 15.77	15.86 15.87 15.32 17.45 14.95 15.85 16.19 17.32 18.50 16.56 15.59 16.84 14.40 15.16	15.74 15.63 14.41 16.61 15.47 14.51 18.79 14.40 17.66 16.43 16.38 16.74 15.61 14.33	15.98 15.50 16.27 15.96 16.23 14.83 16.38 16.86 17.17 15.28 16.68 14.60 16.04 17.46	15.55 16.24 16.22 16.85 16.56 14.38 14.37 15.59 17.22 16.55 16.97 16.88 16.95 15.84	15.77 16.06 15.40 17.20 15.54 15.86 14.79 14.94 16.35 16.25 15.51 15.21 16.95 15.32
(0,-1)	16.94 14.74	16.44 14.36	16.36 14.34	15.18 16.26	14.16 13.95	16.53 13.82	16.07 16.55	14.40 16.28	16.03 14.56	14.11 15.70
(1,-1)	9.93 10.20	11.30 10.82	10.49 10.94	11.05 11.09	11.62 9.72	9.21 8.99	11.27 9.87	11.72 10.28	9.65 12.29	9.43 9.82
(-1,0)	15.77 14.83	15.80 16.84	16.83 16.79	14.93 16.03	16.38 15.29	16.32 15.27	16.99 15.79	15.88 15.78	14.20 14.48	14.62 15.03
(0,0)	13.40 14.12	15.05 16.54	14.31 15.67	15.60 13.59	14.50 13.81	15.00 14.39	12.86 15.84	13.61 12.69	13.51 15.89	14.89 14.28
(1,0)	7.71 9.14	10.61 7.83	8.04 10.34	9.36 9.72	8.13 7.57	9.71 12.09	9.19 8.72	7.00 8.69	8.03 11.62	8.93 10.87
(-1,1)	13.77 13.54	13.88 13.39	12.37 14.82	13.77 13.14	13.40 13.90	13.47 13.55	15.78 15.09	13.39 14.87	12.75 15.60	15.86 13.93
(0,1)	11.92 12.48	10.85 11.24	12.60 12.68	13.39 13.71	11.18 10.66	12.36 13.45	11.22 11.05	12.11 9.62	12.43 11.86	11.06 13.22
(1,1)	6.61 6.74	4.74 5.36	5.61 6.28	7.65 4.91	6.66 5.36	5.58 6.41	6.43 6.12	5.97 5.92	4.14 5.27	5.37 4.94

3. "C" Samples

SAMPLE

Test Design Point	1C	2C	3C	4C	5C	6C	7C	8C	9C	10C
(-1,-1)	16.09	15.21	15.08	16.36	14.48	15.86	15.74	15.98	15.55	15.77
	15.72	16.40	16.57	14.95	16.45	15.87	15.63	15.50	16.24	16.06
	16.98	16.54	14.17	16.94	17.28	15.32	14.41	16.27	16.22	15.40
	16.54	14.55	15.77	17.07	16.87	17.45	16.61	15.96	16.85	17.20
	18.35	14.99	16.00	14.74	16.99	14.95	15.47	16.23	16.56	15.54
	17.86	14.78	16.41	15.38	16.76	15.85	14.51	14.83	14.38	15.86
	15.62	17.06	15.34	17.39	14.76	16.19	18.79	16.38	14.37	14.79
	16.05	17.36	15.61	15.61	15.11	17.32	14.40	16.86	15.59	14.94
	17.23	15.68	15.57	16.05	14.66	18.50	17.66	17.17	17.22	16.35
	17.34	15.18	16.16	13.53	17.18	16.56	16.43	15.28	16.55	16.25
	16.05	15.84	15.80	15.89	15.74	15.59	16.38	16.68	16.97	15.51
	15.45	14.88	15.17	15.91	14.84	16.84	16.74	14.60	16.88	15.21
	15.92	15.73	15.02	15.43	17.16	14.40	15.61	16.04	16.95	16.95
	16.92	13.90	16.46	16.42	15.77	15.16	14.33	17.46	15.84	15.32
	16.48	14.26	16.10	16.43	17.44	15.76	17.52	15.50	15.64	15.34
	16.69	15.26	16.60	16.57	15.05	14.85	16.04	17.28	16.34	16.65
	15.53	15.00	13.68	16.38	16.66	15.62	17.73	15.37	14.87	15.30
	16.28	15.53	18.00	14.69	15.53	14.43	15.85	16.48	15.62	16.09
	16.48	15.98	15.76	17.16	16.55	15.06	16.80	16.87	16.56	17.81
	16.31	16.21	16.97	16.75	15.45	16.53	15.77	17.34	16.43	17.09
	16.30	16.23	15.01	14.87	16.65	14.92	15.02	14.97	14.62	14.85
	16.90	14.87	16.52	16.30	15.72	16.69	15.57	15.57	14.72	16.80
	16.14	15.89	18.75	15.52	15.67	15.84	16.53	14.15	14.58	15.56
	16.56	15.21	15.98	15.96	16.45	17.79	16.95	16.23	17.51	15.63
	15.73	16.23	15.74	17.59	16.61	16.58	16.04	16.50	16.03	14.70
	15.54	16.91	15.45	16.63	16.27	14.49	17.07	16.73	14.95	15.13
	16.04	15.68	14.99	16.38	16.96	15.52	13.87	16.74	15.25	15.19
	17.20	15.62	14.69	16.26	18.47	15.82	16.35	15.99	16.19	17.52

3. "C" Samples (Cont'd)

SAMPLE

Test Design Point	1C	2C	3C	4C	5C	6C	7C	8C	9C	10C
(0,-1)	16.94 14.74 16.94 15.99	16.44 14.36 15.78 15.34	16.36 14.34 15.91 15.78	15.18 16.26 16.44 15.16	14.16 13.95 14.68 14.90	16.53 13.82 14.43 16.70	16.07 16.55 14.70 15.50	14.40 16.28 15.38 15.27	16.03 14.56 16.38 14.03	14.11 15.70 16.16 15.33
(1,-1)	9.93 10.20 9.63 10.99	11.30 10.82 12.18 10.51	10.49 10.94 10.90 10.75	11.05 11.09 9.87 11.33	11.62 9.72 9.54 10.49	9.21 8.99 10.87 9.19	11.27 9.87 9.10 9.54	11.72 10.28 10.45 9.58	9.65 12.29 10.82 10.65	9.43 9.82 11.01 9.72
(-1,0)	15.77 14.83 16.34 16.34	15.80 16.84 15.34 16.09	16.83 16.79 16.49 15.95	14.93 16.03 16.88 15.06	16.38 15.29 15.92 14.78	16.32 15.27 17.07 15.70	16.99 15.79 16.08 14.94	15.88 15.78 16.94 16.41	14.20 14.48 17.00 14.51	14.62 15.03 16.51 15.30
(0,0)	13.40 14.12 15.13 15.45	15.05 16.54 14.19 15.06	14.31 15.67 14.77 14.48	15.60 13.59 14.70 13.90	14.50 13.81 14.09 15.14	15.00 14.39 14.10 14.22	12.86 15.84 15.11 16.46	13.61 12.69 15.25 14.45	13.51 15.89 13.70 15.88	14.89 14.28 14.86 13.64
(1,0)	7.71 9.14 9.31 9.36	10.61 7.83 8.77 9.60	8.04 10.34 8.32 10.46	9.36 9.72 7.75 9.30	8.13 7.57 9.43 9.85	9.71 12.09 11.56 8.03	9.19 8.72 9.04 9.05	7.00 8.69 9.42 8.79	8.03 11.62 8.07 9.15	8.93 10.87 10.49 8.49
(-1,1)	13.77 13.54 15.02 12.65	13.88 13.39 15.14 13.89	12.37 14.82 12.19 14.80	13.77 13.14 15.05 13.08	13.40 13.90 14.52 12.07	13.47 13.55 12.66 13.24	15.78 15.09 13.77 12.87	13.39 14.87 13.94 12.14	12.75 15.60 13.20 12.80	15.86 13.93 13.20 13.80
(0,1)	11.92 12.48 11.58 12.08	10.85 11.24 13.54 12.07	12.60 12.68 12.08 10.60	13.39 13.71 11.71 11.39	11.18 10.66 12.99 12.16	12.36 13.45 10.17 14.27	11.22 11.05 11.33 11.90	12.11 9.62 10.90 10.94	12.43 11.86 12.45 13.59	11.06 13.22 11.94 12.91
(1,1)	6.61 6.74 6.07 8.10	4.74 5.36 6.15 4.76	5.61 6.28 7.57 7.43	7.65 4.91 6.88 7.25	6.66 5.36 5.03 6.12	5.58 6.41 7.56 7.16	6.43 6.12 7.76 4.78	5.97 5.92 6.84 6.26	4.14 5.27 6.04 6.65	5.37 4.94 6.90 6.49

B. Analysis of Responses at Test Point (-1, -1)

1. "A" Samples, N = 15 (7 at (-1, -1))

Sample	$\bar{Y}$	Variance	S	$\bar{Y}$ - KS
1A	16.74	1.11022	1.054	10.35
2A	15.65	0.98978	0.995	9.62
3A	15.62	0.69433	0.833	10.57
4A	16.12	1.17772	1.085	9.56
5A	16.23	1.27372	1.129	9.39
6A	15.93	0.62088	0.788	11.15
7A	15.88	2.21483	1.488	6.86
8A	15.88	0.29798	0.546	12.57
9A	15.74	1.02458	1.012	9.61
10A	15.80	0.54662	0.739	11.32
Averages	15.96	0.99507	0.967	10.10

2. "B" Samples, N = 30 (14 at (-1, -1))

Sample	$\bar{Y}$	Variance	S	$\bar{Y}$ - KS
1B	16.58	0.78648	0.887	12.42
2B	15.58	0.97779	0.989	10.94
3B	15.65	0.44202	0.665	12.53
4B	15.83	1.04261	1.021	11.04
5B	16.00	1.13809	1.067	11.00
6B	16.13	1.22775	1.108	10.93
7B	15.91	1.72342	1.313	9.75
8B	16.09	0.68952	0.830	12.20
9B	16.16	0.83784	0.915	11.87
10B	15.80	0.50072	0.708	12.48
Averages	15.97	0.93662	0.950	11.52

3. "C" Samples, N = 60 (28 at (-1, -1))

Sample	$\bar{Y}$	Variance	S	$\bar{Y}$ - KS
1C	16.44	0.51193	0.715	13.53
2C	15.61	0.69610	0.834	12.22
3C	15.83	1.08686	1.043	11.58
4C	16.04	0.84844	0.921	12.29
5C	16.20	0.97113	0.985	12.19
6C	15.92	1.07163	1.035	11.71
7C	16.07	1.35407	1.164	11.33
8C	16.09	0.72324	0.850	12.63
9C	15.91	0.84103	0.917	12.18
10C	15.87	0.75011	0.866	12.35
Averages	16.00	0.88545	0.933	12.20

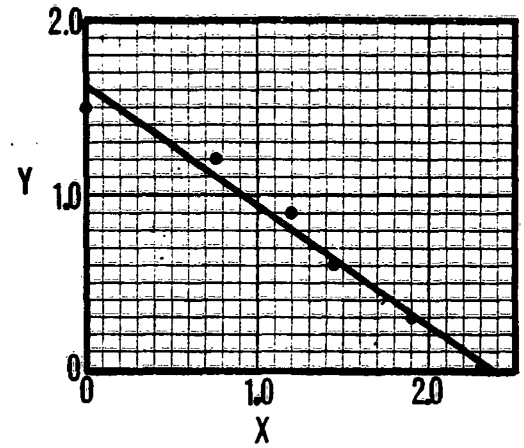
C. Testing the Assumption of Normal Distribution at Point (-1, -1)

The necessary calculations are given for samples 1A, 1B and 1C for 7, 14, and 28 results respectively at (-1, -1). Calculations for the remaining samples are similar. (See Appendix 1B for specific instructions).

	Sample 1A, N = 7			Sample 1B, N = 14			Sample 1C, N = 28		
Y	Freq	Cum Freq	Cum Freq (% of 2N)	Freq	Cum Freq	Cum Freq (% of 2N)	Freq	Cum Freq	Cum Freq (% of 2N)
18.6	0	0	-	0	0	-	0	0	-
18.2	1	1	7.1	1	1	3.6	1	1	1.8
17.8	1	3	21	1	3	11	1	3	5.4
17.4	0	4	29	2	6	21	3	7	13
17.0	1	5	36	2	10	36	3	13	23
16.6	2	8	57	2	14	50	6	22	39
16.2	1	11	79	3	19	68	8	36	64
15.8	1	13	92.9	2	24	86	3	47	84
15.4	0	14	-	1	27	96.4	3	53	94.6
15.0				0	28	-	0	56	-

The cumulative frequency percentages are then plotted against the midpoint values of the Y intervals on arithmetical probability paper as shown on Figures 10, 11, 12 and 13. The least squares straight lines are then drawn through the solid points representing percentages between 10 and 90. In the case of the 10 samples 1A to 10A on Figure 10, a best straight line can be represented only for sample 8A. The wide divergence of the points for the other samples require calculation of the equation for the least squares straight line. For the "B" samples this had to be done for samples 5B, 6B, 7B, 9B and 10B. For the "C" samples it had to be done only for 5C and 10C, the other samples permitting the estimation of the best straight line directly. A technique for deriving the least squares straight line is illustrated for sample 5C.

The solid points for the sample are transferred, from Figure 12, to regular graph paper.



Arbitrary scale values are assigned to the two axes called X and Y for convenience. Values of X and Y for the points are read off the graph. A table of values of X, X^2, Y and XY is formed:

	X	X^2	Y	XY
	0	0	1.50	0
	0.77	0.59	1.20	0.92
	1.20	1.44	0.90	1.08
	1.43	2.05	0.60	0.86
	1.90	3.61	0.30	0.57
	2.29	5.24	0	0
Sums	7.59	12.93	4.5	3.43

The slope of the least squares straight line is determined from the equation:

$$b = \frac{\Sigma XY - \frac{(\Sigma X)(\Sigma Y)}{n}}{\Sigma X^2 - \frac{(\Sigma X)^2}{n}} = \frac{3.43 - \frac{(7.59)(4.5)}{6}}{12.93 - \frac{(7.59)^2}{6}} = \frac{3.43 - 5.69}{12.93 - 9.60} = \frac{-2.26}{3.33} = -0.68$$

$$b_0 = \bar{Y} - b\bar{X} = \frac{\Sigma Y}{n} - b\frac{\Sigma X}{n} = 0.75 + 0.86 = 1.61$$

The fitted equation is therefore $Y = 1.61 - 0.68X$

$$\text{at } X = 0,\ Y = 1.61$$

$$\text{at } Y = 0,\ X = \frac{1.61}{0.68} = 2.37$$

The least squares straight line is plotted through these two points and is then measured off and transferred to the cumulative frequency distribution plot for sample 5C on Figure 12.

D. Analysis of "A" Sample Data

1. Coefficients of Fitted Equations

Sample	ΣX_1Y	ΣX_2Y	ΣX_1^2Y	ΣX_2^2Y	ΣX_1X_2Y	ΣY	$2/3\Sigma Y$
1	-22.03	-11.31	70.53	75.91	-0.35	112.79	75.19
2	-18.68	-13.92	71.98	72.86	-4.79	114.32	76.21
3	-20.68	-11.89	68.96	73.05	-1.63	112.23	74.82
4	-16.76	- 7.54	72.88	77.16	-1.05	117.05	78.03
5	-19.60	-10.77	72.42	73.25	-2.13	112.26	74.84
6	-21.22	-10.26	70.22	73.08	-1.17	114.11	76.07
7	-21.76	- 9.79	75.54	76.65	-4.74	115.69	77.13
8	-20.46	-10.53	69.84	73.47	-3.26	109.96	73.31
9	-20.87	-12.10	64.51	70.74	-2.52	106.48	70.99
10	-22.55	- 7.05	70.01	71.63	-4.12	110.07	73.38

	b_1 $\frac{\Sigma X_1Y}{6}$	b_2 $\frac{\Sigma X_2Y}{6}$	b_{12} $\frac{\Sigma X_1X_2Y}{4}$	b_{11} $\frac{\Sigma X_1^2Y-2/3\Sigma Y}{2}$	b_{22} $\frac{\Sigma X_2^2Y-2/3\Sigma Y}{2}$	b_o $\frac{\Sigma Y}{9}-\frac{2}{3}b_{11}-\frac{2}{3}b_{22}$
1	-3.67	-1.89	-0.09	-2.33	0.36	13.85
2	-3.11	-2.32	-1.20	-2.12	-1.68	15.24
3	-3.45	-1.98	-0.41	-2.93	-0.89	15.02
4	-2.79	-1.26	-0.26	-2.58	-0.44	15.02
5	-3.27	-1.80	-0.53	-1.21	-0.80	13.81
6	-3.54	-1.71	-0.29	-2.93	-1.50	15.63
7	-3.63	-1.63	-1.19	-0.80	-0.24	13.55
8	-3.41	-1.76	-0.82	-1.74	0.08	13.33
9	-3.48	-2.02	-0.63	-3.24	-0.13	14.08
10	-3.76	-1.18	-1.03	-1.69	-0.88	13.94

Fitted Equations

1. $Y = 13.85 - 3.67X_1 - 1.89X_2 - 2.33X_1^2 + 0.36X_2^2 - 0.09X_1X_2$
2. $Y = 15.24 - 3.11X_1 - 2.32X_2 - 2.12X_1^2 - 1.68X_2^2 - 1.20X_1X_2$
3. $Y = 15.02 - 3.45X_1 - 1.98X_2 - 2.93X_1^2 - 0.89X_2^2 - 0.41X_1X_2$
4. $Y = 15.02 - 2.79X_1 - 1.26X_2 - 2.58X_1^2 - 0.44X_2^2 - 0.26X_1X_2$
5. $Y = 13.81 - 3.27X_1 - 1.80X_2 - 1.21X_1^2 - 0.80X_2^2 - 0.53X_1X_2$
6. $Y = 15.63 - 3.54X_1 - 1.71X_2 - 2.93X_1^2 - 1.50X_2^2 - 0.29X_1X_2$
7. $Y = 13.55 - 3.63X_1 - 1.63X_2 - 0.80X_1^2 - 0.24X_2^2 - 1.19X_1X_2$
8. $Y = 13.33 - 3.41X_1 - 1.76X_2 - 1.74X_1^2 + 0.08X_2^2 - 0.82X_1X_2$
9. $Y = 14.08 - 3.48X_1 - 2.02X_2 - 3.24X_1^2 - 0.13X_2^2 - 0.63X_1X_2$
10. $Y = 13.94 - 3.76X_1 - 1.18X_2 - 1.69X_1^2 - 0.88X_2^2 - 1.03X_1X_2$

2. Analysis of Variance

	ΣY^2	d.f.	Due to b_0 $\frac{(\Sigma Y)^2}{9}$	d.f.	Due to Linear Terms*	d.f.	Due to Quadratic Terms **	d.f.	Residual	d.f.
1	1,528.8845	9	1,413.5093	1	102.2062	2	11.1476	3	2.0214	3
2	1,564.4452	9	1,452.1180	1	90.4514	2	20.2938	3	1.5819	3
3	1,517.5897	9	1,399.5081	1	94.8391	2	19.4005	3	3.8420	3
4	1,593.6913	9	1,522.3003	1	56.2916	2	13,9154	3	1.1840	3
5	1,492.5022	9	1,400.2564	1	83.3589	2	5.3265	3	3.5604	3
6	1,562.8033	9	1,446.7880	1	92.5927	2	21.9236	3	1.4990	3
7	1,592.1697	9	1,487.1307	1	94.8903	2	6.9962	3	3.1525	3
8	1,444.8844	9	1,343.4668	1	88.2488	2	8.6902	3	4.4786	3
9	1,380.6790	9	1,259.7767	1	96.9945	2	22.6141	3	1.2937	3
10	1,455.5585	9	1,346.1561	1	93.0342	2	11.4534	3	4.9148	3

	Quadratic Mean Square	Residual Mean Square	$F_{3,3}$ Ratio	Quadratic Terms Significant at 95% Level of Significance
1	3.7159	0.6738	5.51	No ***
2	6.7646	0.5273	12.83	Yes
3	6.4668	1.2807	5.05	No
4	4.6385	0.3947	11.75	Yes
5	1.7755	1.1868	1.50	No
6	7.3079	0.4997	14.62	Yes
7	2.3321	1.0508	2.22	No
8	2.8967	1.4929	1.94	No
9	7.5380	0.4312	17.48	Yes
10	3.8178	1.2726	3.00	No

* Linear Terms: $\frac{(\Sigma X_1 Y)^2}{6} + \frac{(\Sigma X_2 Y)^2}{6}$

** Quadratic Terms: $\frac{(\Sigma X_1^2 Y - 2/3\Sigma Y)^2}{2} + \frac{(X_2^2 Y - 2/3\Sigma Y)^2}{2} + \frac{(\Sigma X_1 X_2 Y)^2}{4}$

*** To be significant, the $F_{.95\ 3,3}$ ratio of 9.28 must be exceeded.

3. Predicted Points from Fitted Equations

X_2	X_1	1)	2)	3)	4)	5)	6)	7)	8)	9)	10)
1	-1	13.75	13.43	13.08	13.79	13.80	13.32	15.70	14.14	12.80	14.98
1	0	12.32	11.24	12.15	13.32	11.21	12.42	11.68	11.65	11.93	11.88
1	1	6.23	4.81	5.36	7.69	6.20	5.66	6.06	5.68	4.58	5.40
0	-1	15.19	16.23	15.54	15.23	15.87	16.24	16.38	15.00	14.32	16.01
0	0	13.85	15.24	15.02	15.02	13.81	15.63	13.55	13.33	14.08	13.94
0	1	7.85	10.01	8.64	9.65	9.33	9.16	9.12	8.18	7.36	8.49
-1	-1	17.35	15.67	16.22	15.79	16.34	16.16	16.58	16.02	15.58	15.28
-1	0	16.10	15.88	16.11	15.84	14.81	15.84	14.94	15.17	15.97	14.24
-1	1	10.19	11.85	10.14	10.73	10.86	9.66	11.70	10.84	9.88	9.82

4. Standard Errors

1)	Y	16.10	10.19	15.19	13.85	7.85	13.75	12.32	6.23
	y	16.94	9.93	15.77	13.40	7.71	13.77	11.92	6.61
	(Y-y)	0.84	0.26	0.58	0.45	0.14	0.02	0.40	0.38

$$S=\sqrt{\frac{\Sigma(Y-y)^2}{N-5}}=\sqrt{\frac{1.6365}{3}}=\sqrt{0.54550}=0.739$$

2)(Y-y)	0.56	0.55	0.43	0.19	0.60	0.45	0.39	0.07

$$S=\sqrt{\frac{1.5566}{3}}=\sqrt{0.51887}=0.720$$

3)(Y-y)	0.25	0.35	1.29	0.71	0.60	0.71	0.45	0.25

$$S=\sqrt{\frac{3.4823}{3}}=\sqrt{1.16077}=1.077$$

4. Standard Errors (continued)

4) (Y-y) | 0.66 0.32 0.30 0.58 0.29 0.02 0.07 0.04

$S = \sqrt{\frac{1.0554}{3}} = \sqrt{0.35180} = 0.593$

5) (Y-y) | 0.65 0.76 0.51 0.69 1.20 0.40 0.03 0.46

$S = \sqrt{\frac{3.5488}{3}} = \sqrt{1.18293} = 1.088$

6) (Y-y) | 0.69 0.45 0.08 0.63 0.55 0.15 0.06 0.08

$S = \sqrt{\frac{1.4169}{3}} = \sqrt{0.47230} = 0.687$

7) (Y-y) | 1.13 0.43 0.61 0.69 0.07 0.08 0.46 0.37

$S = \sqrt{\frac{2.6698}{3}} = \sqrt{0.88993} = 0.943$

8) (Y-y) | 0.77 0.88 0.88 0.28 1.18 0.75 0.46 0.29

$S = \sqrt{\frac{4.4707}{3}} = \sqrt{1.49023} = 1.221$

9) (Y-y) | 0.06 0.23 0.12 0.57 0.67 0.05 0.50 0.44

$S = \sqrt{\frac{1.2908}{3}} = \sqrt{0.43027} = 0.656$

10) (Y-y) | 0.13 0.39 1.39 0.95 0.44 0.88 0.82 0.03

$S = \sqrt{\frac{4.6449}{3}} = \sqrt{1.54830} = 1.244$

5. Comparison of Standard Deviations and Standard Errors

Sample	Std. Error	Std. Dev.	Variance (Std. Error)^2	Variance (Std. Dev.)^2	Variance Ratio	*	**Sp^2	Sp
1	0.739	1.054	0.54550	1.11022	0.49	No	0.92198	0.960
2	0.720	0.995	0.51887	0.98978	0.52	No	0.83281	0.913
3	1.077	0.883	1.16077	0.69433	1.67	No	0.84981	0.922
4	0.593	1.085	0.35180	1.17772	0.30	No	0.90241	0.950
5	1.088	1.129	1.18293	1.27372	0.93	No	1.24346	1.115
6	0.687	0.788	0.47230	0.62088	0.76	No	0.57135	0.756
7	0.943	1.488	0.88993	2.21483	0.40	No	1.77320	1.332
8	1.221	0.546	1.49023	0.29798	5.00	No	0.69540	0.834
9	0.656	1.012	0.43027	1.02458	0.42	No	0.82648	0.909
10	1.244	0.739	1.54830	0.54662	2.83	No	0.88051	0.938

* F Ratio Test for Comparison of Variance

Reject the hypothesis that the variances are equal, at the 95% level of significance, if the variance ratio falls outside the region of 6.60 to 0.068

$$\left(F_{.975(3,6\text{ d.f.})} = 6.60, \quad \frac{1}{F_{.975(6,3\text{ d.f.})}} = 0.068 \right)$$

No means no significant difference, permitting pooling of the variances.

** Calculations for Pooling Variance, Sample 1

$$Sp^2 = \frac{3(\text{d.f.}) \times 0.54550 + 6(\text{d.f.}) \times 1.11022}{9\ (\text{d.f.})} = 0.92198$$

$$Sp = \sqrt{0.92198} = 0.960$$

6. Reliability Boundary Equations

(Based on $\bar{Y}$-KS where K = 5.414 for N = 15-6 =9 and for reliability standards of 99.9% at 95% confidence.)

1. $3.67X_1 + 1.89X_2 + 2.33X_1^2 - 0.36X_2^2 + 0.09X_1X_2 = 13.85 - 5.20 - 10.00 = -1.35$
2. $3.11X_1 + 2.32X_2 + 2.12X_1^2 + 1.68X_2^2 + 1.20X_1X_2 = 15.24 - 4.94 - 10.00 = 0.30$
3. $3.45X_1 + 1.98X_2 + 2.93X_1^2 + 0.89X_2^2 + 0.41X_1X_2 = 15.02 - 4.99 - 10.00 = 0.03$
4. $2.79X_1 + 1.26X_2 + 2.58X_1^2 + 0.44X_2^2 + 0.26X_1X_2 = 15.02 - 5.14 - 10.00 = -0.12$
5. $3.27X_1 + 1.80X_2 + 1.21X_1^2 + 0.80X_2^2 + 0.53X_1X_2 = 13.81 - 6.04 - 10.00 = -2.23$
6. $3.54X_1 + 1.71X_2 + 2.93X_1^2 + 1.50X_2^2 + 0.29X_1X_2 = 15.63 - 4.09 - 10.00 = 1.54$
7. $3.63X_1 + 1.63X_2 + 0.80X_1^2 + 0.24X_2^2 + 1.19X_1X_2 = 13.55 - 7.21 - 10.00 = -3.66$
8. $3.41X_1 + 1.76X_2 + 1.74X_1^2 - 0.08X_2^2 + 0.82X_1X_2 = 13.33 - 4.52 - 10.00 = -1.19$
9. $3.48X_1 + 2.02X_2 + 3.24X_1^2 + 0.13X_2^2 + 0.63X_1X_2 = 14.08 - 4.92 - 10.00 = -0.84$
10. $3.76X_1 + 1.18X_2 + 1.69X_1^2 + 0.88X_2^2 + 1.03X_1X_2 = 13.94 - 5.08 - 10.00 = -1.14$

"A" Sample Reliability Boundary Equation for Infinite Number of Samples

$$3.24X_1 + 1.69X_2 + 2.13X_1^2 + 0.94X_2^2 + 0.43X_1X_2 = 14.61 - 5.41 - 10.00 = -0.80$$

E. Analysis of "B" Sample Data

1. Coefficients of Fitted Equations for "B" Samples

Sample	ΣX_1Y	ΣX_2Y	ΣX_1^2Y	ΣX_2^2Y	ΣX_1X_2Y	ΣY	$2/3\Sigma Y$
1	-40.74	-19.91	141.40	150.03	-0.93	225.00	150.00
2	-40.41	-24.62	141.73	143.54	-8.13	226.21	150.81
3	-40.41	-19.07	143.81	147.79	-5.43	229.77	153.18
4	-35.75	-18.67	143.31	151.81	-4.83	231.04	154.03
5	-41.91	-20.29	140.03	142.61	-4.62	218.29	145.53
6	-38.88	-15.99	142.86	145.63	-0.97	228.41	152.27
7	-43.87	-19.89	147.07	151.27	-7.64	230.66	153.77
8	-42.52	-22.98	141.68	146.74	-6.19	220.39	146.93
9	-38.35	-22.80	140.35	146.90	-8.56	224.63	149.75
10	-41.68	-16.28	140.40	145.04	-7.13	223.66	149.11

	b_1 $\frac{\Sigma X_1Y}{12}$	b_2 $\frac{\Sigma X_2Y}{12}$	b_{12} $\frac{\Sigma X_1X_2Y}{8}$	b_{11} $\frac{\Sigma X_1^2Y - 2/3\Sigma Y}{4}$	b_{22} $\frac{\Sigma X_2^2Y - 2/3\Sigma Y}{4}$	b_0 $\frac{\Sigma Y}{18} - 2/3b_{11} - 2/3b_{22}$
1	-3.40	-1.66	-0.12	-2.15	0.01	13.93
2	-3.37	-2.05	-1.02	-2.27	-1.82	15.29
3	-3.37	-1.59	-0.68	-2.34	-1.35	15.23
4	-2.98	-1.56	-0.60	-2.68	-0.56	14.99
5	-3.49	-1.69	-0.58	-1.38	-0.73	13.53
6	-3.24	-1.33	-0.12	-2.35	-1.66	15.37
7	-3.66	-1.66	-0.96	-1.68	-0.63	14.35
8	-3.54	-1.92	-0.77	-1.31	-0.05	13.15
9	-3.20	-1.90	-1.07	-2.35	-0.71	14.52
10	-3.47	-1.36	-0.89	-2.18	-1.02	14.56

Fitted Equations

1. $Y = 13.93 - 3.40X_1 - 1.66X_2 - 2.15X_1^2 + 0.01X_2^2 - 0.12X_1X_2$
2. $Y = 15.29 - 3.37X_1 - 2.05X_2 - 2.27X_1^2 - 1.82X_2^2 - 1.02X_1X_2$
3. $Y = 15.23 - 3.37X_1 - 1.59X_2 - 2.34X_1^2 - 1.35X_2^2 - 0.68X_1X_2$
4. $Y = 14.99 - 2.98X_1 - 1.56X_2 - 2.68X_1^2 - 0.56X_2^2 - 0.60X_1X_2$
5. $Y = 13.53 - 3.49X_1 - 1.69X_2 - 1.38X_1^2 - 0.73X_2^2 - 0.58X_1X_2$
6. $Y = 15.37 - 3.24X_1 - 1.33X_2 - 2.35X_1^2 - 1.66X_2^2 - 0.12X_1X_2$
7. $Y = 14.35 - 3.66X_1 - 1.66X_2 - 1.68X_1^2 - 0.63X_2^2 - 0.96X_1X_2$
8. $Y = 13.15 - 3.54X_1 - 1.92X_2 - 1.31X_1^2 - 0.05X_2^2 - 0.77X_1X_2$
9. $Y = 14.52 - 3.20X_1 - 1.90X_2 - 2.35X_1^2 - 0.71X_2^2 - 1.07X_1X_2$
10. $Y = 14.56 - 3.47X_1 - 1.36X_2 - 2.18X_1^2 - 1.02X_2^2 - 0.89X_1X_2$

2. Analysis of Variance

Sample	ΣY^2	d.f.	Due to b_0 $\frac{(\Sigma Y)^2}{9}$	d.f.	* Due to Linear Terms	d.f.	** Due to Quadratic Terms	d.f.
1	3,007.1098	18	2,812.5000	1	171.3463	2	18.5983	3
2	3,081.1029	18	2,842.8313	1	186.5927	2	42.0869	3
3	3,142.9913	18	2,933.0141	1	166.3861	2	32.8978	3
4	3,143.2496	18	2,965.5268	1	135.5526	2	32.8778	3
5	2,847.5505	18	2,647.2513	1	180.6777	2	12.3622	3
6	3,093.2109	18	2,898.3960	1	147.2779	2	33.2770	3
7	3,181.1040	18	2,955.7798	1	193.3491	2	20.0812	3
8	2,915.4713	18	2,698.4307	1	194.6692	2	11.6891	3
9	3,027.2397	18	2,803.2576	1	165.8802	2	33.2798	3
10	2,989.3140	18	2,779.0998	1	166.8550	2	29.4618	3

Sample	Residual	d.f.	Experimental Error	d.f.	Lack of Fit	d.f.	Quadratic Mean Square	Residual Mean Square
1	4.6652	12	4.3716	8	0.2936	4	6.1994	0.3888
2	9.5920	12	8.1818	8	1.4102	4	14.0290	0.7993
3	10.6933	12	8.9410	8	1.7523	4	10.9659	0.8911
4	9.2924	12	7.2773	8	2.0151	4	10.9593	0.7744
5	7.2593	12	3.9212	8	3.3381	4	4.1207	0.6049
6	14.2600	12	8.2075	8	6.0525	4	11.0923	1.1883
7	11.8939	12	6.6664	8	5.2275	4	6.6937	0.9912
8	10.6823	12	8.8568	8	1.8255	4	3.8964	0.8902
9	24.8221	12	18.7429	8	6.0792	4	11.0933	2.0685
10	13.8974	12	7.7797	8	6.1177	4	9.8206	1.1581

Sample	$F_{3,12}$ Ratio	***	Lack of Fit Mean Square	Experimental Error Mean Square	$F_{4,8}$ Ratio	****
1	15.94	Yes	0.0734	0.5465	0.13	No
2	17.55	Yes	0.3526	1.0227	0.34	No
3	12.31	Yes	0.4381	1.1176	0.39	No
4	14.15	Yes	0.5038	0.9097	0.55	No
5	6.81	Yes	0.8345	0.4902	1.70	No
6	9.33	Yes	1.5131	1.0259	1.47	No
7	6.75	Yes	1.3069	0.8333	1.57	No
8	4.38	Yes	0.4564	1.1071	0.41	No
9	5.36	Yes	1.5198	2.3429	0.65	No
10	8.48	Yes	1.5294	0.9725	1.57	No

*Linear Terms: $\frac{(\Sigma X_1 Y)^2}{12} + \frac{(\Sigma X_2 Y)^2}{12}$

**Quadratic Terms: $\frac{(\Sigma X_1^2 Y - 2/3 \Sigma Y)^2}{4} + \frac{(\Sigma X_2^2 Y - 2/3 \Sigma Y)^2}{4} + \frac{(\Sigma X_1 X_2 Y)^2}{8}$

***Are quadratic terms significant at the 95% level of significance? To be significant the $F_{3,12}$ ratio must exceed $F_{.95\ 3,12} = 3.49$

****Is the lack of fit term significant, at the 95% level of significance, in comparison to the experimental error? To be significant the $F_{4,8}$ ratio must exceed $F_{.95\ 4,8} = 3.84$.

3. Predicted Points from "B" Sample Fitted Equations

X_2 = 1

Point	X_1 = -1	X_1 = 0	X_1 = 1
1)	13.65	12.28	6.61
2)	13.54	11.42	4.76
3)	14.00	12.29	5.90
4)	13.77	12.87	6.61
5)	13.80	11.11	5.66
6)	13.39	12.38	6.67
7)	15.00	12.06	5.76
8)	14.18	11.18	5.56
9)	13.83	11.91	5.29
10)	14.36	12.18	5.64

X_2 = 0

Point	X_1 = -1	X_1 = 0	X_1 = 1
1)	15.18	13.93	8.38
2)	16.39	15.29	9.65
3)	16.26	15.23	9.52
4)	15.29	14.99	9.33
5)	15.64	13.53	8.66
6)	16.26	15.37	9.78
7)	16.33	14.35	9.01
8)	15.38	13.15	8.30
9)	15.37	14.52	8.97
10)	15.85	14.56	8.91

X_2 = -1

Point	X_1 = -1	X_1 = 0	X_1 = 1
1)	16.73	15.60	10.17
2)	15.60	15.52	10.90
3)	15.82	15.47	10.44
4)	15.69	15.99	10.93
5)	16.02	14.49	10.20
6)	15.81	15.04	9.57
7)	16.40	15.38	11.00
8)	16.48	15.02	10.94
9)	15.49	15.71	11.23
10)	15.30	14.90	10.14

4. "B" Sample Standard Errors

1)

Y	15.60	10.17	15.18	13.93	8.38	13.65	12.28	6.61
y	16.94	9.93	15.77	13.40	7.71	13.77	11.92	6.61
	14.74	10.20	14.83	14.12	9.14	13.54	12.48	6.74
(Y-y)	1.34	0.24	0.59	0.53	0.67	0.12	0.36	0.00
	0.86	0.03	0.35	0.19	0.76	0.11	0.20	0.13

$$S = \sqrt{\frac{\Sigma(Y-y)^2}{N-5}} = \sqrt{\frac{4.6208}{11}} = \sqrt{0.42007} = 0.648$$

2)

(Y-y)	0.92	0.40	0.59	0.24	0.96	0.34	0.57	0.02
	1.16	0.08	0.45	1.25	1.82	0.15	0.18	0.60

$$S = \sqrt{\frac{9.6189}{11}} = \sqrt{0.87445} = 0.935$$

3) (Y-y)

0.89	0.05	0.57	0.92	1.48	1.63	0.31	0.29
1.13	0.50	0.53	0.44	0.82	0.82	0.39	0.38

$$S = \sqrt{\frac{10.6361}{11}} = \sqrt{0.96692} = 0.983$$

4) (Y-y)

0.81	0.12	0.36	0.61	0.03	0.00	0.52	1.04
0.27	0.16	0.74	1.40	0.39	0.63	0.84	1.70

$$S = \sqrt{\frac{9.2758}{11}} = \sqrt{0.84325} = 0.918$$

5) (Y-y)

0.33	1.42	0.74	0.97	0.53	0.40	0.07	1.00
0.54	0.48	0.35	0.28	1.09	0.10	0.45	0.30

$$S = \sqrt{\frac{7.2731}{11}} = \sqrt{0.66119} = 0.813$$

6) (Y-y)

1.49	0.36	0.06	0.37	0.07	0.08	0.02	1.09
1.22	0.58	0.99	0.98	2.31	0.16	1.07	0.26

$$S = \sqrt{\frac{14.0295}{11}} = \sqrt{1.27541} = 1.129$$

7) (Y-y)

0.69	0.27	0.66	1.49	0.18	0.78	0.84	0.67
1.17	1.13	0.54	1.49	0.29	0.09	1.01	0.36

$$S = \sqrt{\frac{11.3994}{11}} = \sqrt{1.03631} = 1.018$$

8) (Y-y)

0.62	0.78	0.50	0.46	1.30	0.79	0.93	0.41
1.26	0.66	0.40	0.46	0.39	0.69	1.56	0.36

$$S = \sqrt{\frac{10.3877}{11}} = \sqrt{0.94434} = 0.972$$

9) (Y-y)

0.32	1.58	1.17	1.01	0.94	1.08	0.52	1.15
1.15	1.06	0.89	1.37	2.65	1.77	0.05	0.02

$$S = \sqrt{\frac{23.9041}{11}} = \sqrt{2.17310} = 1.474$$

10) (Y-y)

0.79	0.71	1.23	0.33	0.02	1.50	1.12	0.27
0.80	0.32	0.82	0.28	1.96	0.43	1.04	0.70

$$S = \sqrt{\frac{13.4190}{11}} = \sqrt{1.21991} = 1.104$$

5. Comparison of "B" Sample Standard Deviations and Standard Errors

Sample	Std. Error	Std. Dev.	-- Variance -- $(\text{Std.Error})^2$	$(\text{Std.Dev.})^2$	Variance Ratio	*	S_p^2	S_p
1	0.648	0.887	0.42007	0.78648	0.53	No	0.61854	0.786
2	0.935	0.989	0.87445	0.97779	0.89	No	0.93043	0.965
3	0.983	0.665	0.96692	0.44202	2.19	No	0.68260	0.826
4	0.918	1.021	0.84325	1.04261	0.81	No	0.95124	0.975
5	0.813	1.067	0.66119	1.13809	0.58	No	0.91951	0.959
6	1.129	1.108	1.27541	1.22775	1.04	No	1.24959	1.118
7	1.018	1.313	1.03631	1.72342	0.60	No	1.40849	1.187
8	0.972	0.830	0.94434	0.68952	1.37	No	0.80631	0.898
9	1.474	0.915	2.17310	0.83784	2.59	No	1.44983	1.204
10	1.104	0.708	1.21991	0.50072	2.44	No	0.83035	0.911

* F Ratio Test for Comparison of Variance

Reject the hypothesis that the variances are equal, at the 95% level of significance, if the variance ratio falls outside the region of 3.20 to 0.293 $\left(F_{.975(11,13\ d.f.)} = 3.20\ ,\ \dfrac{1}{F_{.975(13,11\ d.f.)}} = 0.293\right)$. No means no significant difference, permitting pooling of the variances.

6. "B" Sample Reliability Boundary Equations

(Based on $\bar{\bar{Y}}$ - KS where K = 4.171 for N = 30 - 6 = 24 and for reliability standards of 99.9% at 95% confidence.)

1. $3.40X_1 + 1.66X_2 + 2.15X_1^2 - 0.01X_2^2 + 0.12X_1X_2 = 3.93 - 3.28 - 10.00 = 0.65$
2. $3.37X_1 + 2.05X_2 + 2.27X_1^2 + 1.82X_2^2 + 1.02X_1X^2 = 15.29 - 4.03 - 10.00 = 1.26$
3. $3.37X_1 + 1.59X_2 + 2.34X_1^2 + 1.35X_2^2 + 0.68X_1X_2 = 15.23 - 3.45 - 10.00 = 1.78$
4. $2.98X_1 + 1.56X_2 + 2.68X_1^2 + 0.56X_2^2 + 0.60X_1X_2 = 14.99 - 4.07 - 10.00 = 0.92$
5. $3.49X_1 + 1.69X_2 + 1.38X_1^2 + 0.73X_2^2 + 0.58X_1X_2 = 13.53 - 4.00 - 10.00 = -0.47$
6. $3.24X_1 + 1.33X_2 + 2.35X_1^2 + 1.66X_2^2 + 0.12X_1X_2 = 15.37 - 4.66 - 10.00 = 0.71$
7. $3.66X_1 + 1.66X_2 + 1.68X_1^2 + 0.63X_2^2 + 0.96X_1X_2 = 14.35 - 4.95 - 10.00 = -0.60$
8. $3.54X_1 + 1.92X_2 + 1.31X_1^2 + 0.05X_2^2 + 0.77X_1X_2 = 13.15 - 3.75 - 10.00 = -0.60$
9. $3.20X_1 + 1.90X_2 + 2.35X_1^2 + 0.71X_2^2 + 1.07X_1X_2 = 14.52 - 5.02 - 10.00 = -0.50$
10. $3.47X_1 + 1.36X_2 + 2.18X_1^2 + 1.02X_2^2 + 0.89X_1X_2 = 14.56 - 3.80 - 10.00 = 0.76$

"B" Sample Reliability Boundary Equation for Infinite Number of Samples

$$3.24X_1 + 1.69X_2 + 2.13X_1^2 + 0.94X_2^2 + 0.43X_1X_2 = 14.61 - 4.17 - 10.00 = 0.44$$

F. Analysis of "C" Sample Data

1. Coefficients of Fitted Equations

Sample	$\Sigma X_1 Y$	$\Sigma X_2 Y$	$\Sigma X_1{}^2 Y$	$\Sigma X_1{}^2 Y$	$\Sigma X_1 X_2 Y$	ΣY	$2/3\,\Sigma Y$
1	-80.23	-40.56	287.81	301.68	- 2.45	458.58	305.72
2	-80.18	-44.16	285.44	294.18	-17.66	455.90	303.93
3	-76.43	-39.76	290.69	297.82	- 7.05	460.27	306.85
4	-75.94	-38.61	288.26	302.47	- 7.53	459.29	306.19
5	-81.54	-39.81	280.58	287.91	- 7.29	442.80	295.20
6	-74.60	-33.54	287.32	293.30	- 0.79	456.76	304.51
7	-84.72	-38.78	286.46	294.98	- 7.92	455.05	303.37
8	-82.79	-44.82	284.63	290.62	- 7.02	445.53	297.02
9	-75.80	-41.27	280.56	294.83	-12.02	450.87	300.58
10	-79.27	-35.14	284.19	294.38	-9.59	452.29	301.53

Sample	$\frac{\Sigma X_1 Y}{24}$	$\frac{\Sigma X_2 Y}{24}$	$\frac{\Sigma X_1 X_2 Y}{16}$	$\frac{\Sigma X_1{}^2 Y - \frac{2}{3}\Sigma Y}{8}$	$\frac{\Sigma X_2{}^2 Y - \frac{2}{3}\Sigma Y}{8}$	$\frac{\Sigma Y}{36} - \frac{2}{3} b_{11} - \frac{2}{3} b_{22}$
1	-3.34	-1.69	-0.15	-2.24	-0.51	14.57
2	-3.34	-1.84	-1.10	-2.31	-1.22	15.02
3	-3.18	-1.66	-0.44	-2.02	-1.13	14.88
4	-3.16	-1.61	-0.47	-2.24	-0.47	14.56
5	-3.40	-1.66	-0.46	-1.83	-0.91	14.13
6	-3.11	-1.40	-0.05	-2.15	-1.40	15.06
7	-3.53	-1.62	-0.50	-2.11	-1.05	14.75
8	-3.45	-1.87	-0.44	-1.55	-0.80	13.94
9	-3.16	-1.72	-0.75	-2.50	-0.72	14.67
10	-3.30	-1.46	-0.60	-2.17	-0.89	14.61

Fitted Equations

1. $Y = 14.57 - 3.34X_1 - 1.69X_2 - 2.24X_1{}^2 - 0.51X_2{}^2 - 0.15X_1X_2$
2. $Y = 15.02 - 3.34X_1 - 1.84X_2 - 2.31X_1{}^2 - 1.22X_2{}^2 - 1.10X_1X_2$
3. $Y = 14.88 - 3.18X_1 - 1.66X_2 - 2.02X_1{}^2 - 1.13X_2{}^2 - 0.44X_1X_2$
4. $Y = 14.56 - 3.16X_1 - 1.61X_2 - 2.24X_1{}^2 - 0.47X_2{}^2 - 0.47X_1X_2$
5. $Y = 14.13 - 3.40X_1 - 1.66X_2 - 1.83X_1{}^2 - 0.91X_2{}^2 - 0.46X_1X_2$
6. $Y = 15.06 - 3.11X_1 - 1.40X_2 - 2.15X_1{}^2 - 1.40X_2{}^2 - 0.05X_1X_2$
7. $Y = 14.75 - 3.53X_1 - 1.62X_2 - 2.11X_1{}^2 - 1.05X_2{}^2 - 0.50X_1X_2$
8. $Y = 13.94 - 3.45X_1 - 1.87X_2 - 1.55X_1{}^2 - 0.80X_2{}^2 - 0.44X_1X_2$
9. $Y = 14.67 - 3.16X_1 - 1.72X_2 - 2.50X_1{}^2 - 0.72X_2{}^2 - 0.75X_1X_2$
10. $Y = 14.61 - 3.30X_1 - 1.46X_2 - 2.17X_1{}^2 - 0.89X_2{}^2 - 0.60X_1X_2$

2. Predicted Points from "C" Sample Fitted Equations

X_2	X_1 = -1		X_1 = 0		X_1 = 1	
1	1) 13.64	6) 13.26	1) 12.38	6) 12.25	1) 6.64	6) 5.94
	2) 14.09	7) 14.00	2) 11.96	7) 12.08	2) 5.19	7) 5.94
	3) 13.71	8) 13.62	3) 12.10	8) 11.27	3) 6.45	8) 5.84
	4) 13.88	9) 13.64	4) 12.48	9) 12.33	4) 6.60	9) 5.82
	5) 13.58	10) 13.98	5) 11.55	10) 12.24	5) 5.86	10) 6.18
0	1) 15.67	6) 16.01	1) 14.57	6) 15.05	1) 8.99	6) 9.79
	2) 16.04	7) 16.16	2) 15.02	7) 14.74	2) 9.36	7) 9.10
	3) 16.06	8) 15.85	3) 14.89	8) 13.94	3) 9.68	8) 8.95
	4) 15.49	9) 15.33	4) 14.56	9) 14.67	4) 9.15	9) 9.01
	5) 15.70	10) 15.74	5) 14.13	10) 14.60	5) 8.90	10) 9.14
-1	1) 16.70	6) 15.96	1) 15.76	6) 15.05	1) 10.34	6) 9.84
	2) 15.55	7) 16.24	2) 15.64	7) 15.32	2) 11.09	7) 10.18
	3) 16.15	8) 16.48	3) 15.42	8) 15.01	3) 10.65	8) 10.46
	4) 16.16	9) 15.58	4) 15.70	9) 15.67	4) 10.76	9) 10.76
	5) 15.98	10) 15.72	5) 14.87	10) 15.18	5) 10.10	10) 10.32

3. "C" Sample Standard Errors

1) Y	15.76	10.34	15.67	14.57	8.99	13.64	12.38	6.64
	16.94	9.93	15.77	13.40	7.71	13.77	11.92	6.61
y	14.74	10.20	14.83	14.12	9.14	13.54	12.48	6.74
	16.94	9.63	16.34	15.13	9.31	15.02	11.58	6.07
	15.99	10.99	16.34	15.45	9.36	12.65	12.08	8.10
	1.18	0.41	0.10	1.17	1.28	0.13	0.46	0.03
(Y-y)	1.02	0.14	0.84	0.45	0.15	0.10	0.10	0.10
	1.18	0.71	0.67	0.56	0.32	1.38	0.80	0.57
	0.23	0.65	0.67	0.88	0.37	0.99	0.30	1.46

$$S = \sqrt{\frac{(Y-y)^2}{N-5}} = \sqrt{\frac{17.4958}{27}} = \sqrt{0.64799} = 0.805$$

3. "C" Sample Standard Errors (continued)

2)(Y-y)

0.80	0.21	0.24	0.03	1.25	0.21	1.11	0.45
1.28	0.27	0.80	1.52	1.53	0.70	0.72	0.17
0.14	1.09	0.70	0.83	0.59	1.05	1.58	0.96
0.30	0.58	0.05	0.04	0.24	0.20	0.11	0.43

$$S = \sqrt{\frac{19.8040}{27}} = \sqrt{0.73348} = 0.856$$

3)(Y-y)

0.94	0.16	0.77	0.58	1.64	1.34	0.50	0.84
1.08	0.29	0.73	0.78	0.66	1.11	0.58	0.17
0.49	0.25	0.43	0.12	1.36	1.52	0.02	1.12
0.36	0.10	0.11	0.41	0.78	1.09	1.50	0.98

$$S = \sqrt{\frac{22.9475}{27}} = \sqrt{0.84991} = 0.922$$

4)(Y-y)

0.52	0.29	0.56	1.04	0.21	0.11	0.91	1.05
0.56	0.33	0.54	0.97	0.57	0.74	1.23	1.69
0.74	0.89	1.39	0.14	1.40	1.17	0.77	0.28
0.54	0.57	0.43	0.66	0.15	0.80	1.09	0.65

$$S = \sqrt{\frac{21.4347}{27}} = \sqrt{0.79388} = 0.891$$

5)(Y-y)

0.71	1.52	0.68	0.37	0.77	0.18	0.37	0.80
0.92	0.38	0.41	0.32	1.33	0.32	0.89	0.50
0.19	0.56	0.22	0.04	0.53	0.94	1.44	0.83
0.03	0.39	0.92	1.01	0.95	1.51	0.61	0.26

$$S = \sqrt{\frac{18.9592}{27}} = \sqrt{0.70219} = 0.838$$

6)(Y-y)

1.48	0.63	0.31	0.05	0.08	0.21	0.11	1.36
1.23	0.85	0.74	0.66	2.30	0.29	1.20	0.53
0.62	1.03	1.06	0.95	1.77	0.60	2.08	0.62
1.65	0.65	0.31	0.83	1.76	0.02	2.02	0.22

$$S = \sqrt{\frac{37.7436}{27}} = \sqrt{1.39791} = 1.183$$

3. "C" Sample Standard Errors (continued)

7) (Y-y)	0.75	1.09	0.83	1.88	0.09	1.78	0.86	0.49
	1.23	0.31	0.37	1.10	0.38	1.09	1.03	0.18
	0.62	1.08	0.08	0.37	0.06	0.23	0.75	1.82
	0.18	0.64	1.22	1.72	0.05	1.13	0.18	1.16

$$S = \sqrt{\frac{28.6835}{27}} = \sqrt{1.06235} = 1.031$$

8) (Y-y)	0.61	1.26	0.03	0.33	1.95	0.23	0.84	0.13
	1.27	0.18	0.07	1.25	0.26	1.25	1.65	0.08
	0.37	0.01	1.09	1.31	0.47	0.32	0.37	1.00
	0.26	0.88	0.56	0.51	0.16	1.48	0.33	0.42

$$S = \sqrt{\frac{22.6375}{27}} = \sqrt{0.83843} = 0.916$$

9) (Y-y)	0.36	1.11	1.13	1.16	0.98	0.89	0.20	1.68
	1.11	1.53	0.85	1.22	2.61	1.96	0.37	0.55
	0.71	0.06	1.67	0.97	0.94	0.44	0.22	0.22
	1.64	0.11	0.82	1.21	0.14	0.84	1.36	0.83

$$S = \sqrt{\frac{38.9895}{27}} = \sqrt{1.44398} = 1.202$$

10) (Y-y)	1.07	0.89	1.12	0.29	0.21	1.88	1.18	0.81
	0.52	0.50	0.71	0.32	1.73	0.05	0.98	1.24
	0.98	0.69	0.77	0.26	1.35	0.78	0.30	0.72
	0.15	0.60	0.44	0.96	0.65	0.18	0.67	0.31

$$S = \sqrt{\frac{23.1567}{27}} = \sqrt{0.85766} = 0.926$$

4. Comparison of "C" Sample Standard Deviations and Standard Errors

Sample	Std. Error	Std. Dev.	Variance (Std. Error)^2	Variance (Std. Dev.)^2	Variance Ratio	*	Sp^2	Sp
1	0.805	0.715	0.64799	0.51193	1.27	No	0.57996	0.762
2	0.356	0.834	0.73348	0.69610	1.05	No	0.71479	0.845
3	0.922	1.043	0.84991	1.08686	0.78	No	0.96839	0.984
4	0.891	0.921	0.79388	0.84844	0.94	No	0.82116	0.906
5	0.838	0.985	0.70219	0.97113	0.72	No	0.83666	0.915
6	1.183	1.035	1.39791	1.07163	1.30	No	1.23477	1.111
7	1.031	1.164	1.06235	1.35407	0.78	No	1.20821	1.099
8	0.916	0.850	0.83843	0.72324	1.16	No	0.78109	0.884
9	1.202	0.917	1.44398	0.84103	1.72	No	1.14251	1.069
10	0.926	0.866	0.85766	0.75011	1.14	No	0.80389	0.897

* F Ratio Test for Comparison of Variance

Reject the hypothesis that the variances are equal, at the 95% level of significance, if the variance ratio falls outside the region of 2.17 to 0.461

$$\left((F_{.975(27,27\ d.f.)} + 2.17, \quad \frac{1}{F_{.975(27,27\ d.f.)}} = 0.461) \right)$$

No means no significant difference, permitting pooling of the variances.

5. "C" Sample Reliability Boundary Equations

(Based on Y-KS where K= 3.731 for N= 60 - 6 = 54 and for reliability standards of 99.9% at 95% confidence.)

1. $3.34X_1 + 1.69X_2 + 2.24X_1^2 + 0.51X_2^2 + 0.15X_1X_2 = 14.57 - 2.84 - 10.00 = 1.73$

2. $3.34X_1 + 1.84X_2 + 2.31X_1^2 + 1.22X_2^2 + 1.10X_1X_2 = 15.02 - 3.15 - 10.00 = 1.87$

3. $3.18X_1 + 1.66X_2 + 2.02X_1^2 + 1.13X_2^2 + 0.44X_1X_2 = 14.88 - 3.67 - 10.00 = 1.21$

4. $3.16X_1 + 1.61X_2 + 2.24X_1^2 + 0.47X_2^2 + 0.47X_1X_2 = 14.56 - 3.38 - 10.00 = 1.18$

5. $3.40X_1 + 1.66X_2 + 1.83X_1^2 + 0.91X_2^2 + 0.46X_1X_2 = 14.13 - 3.41 - 10.00 = 0.72$

6. $3.11X_1 + 1.40X_2 + 2.15X_1^2 + 1.40X_2^2 + 0.05X_1X_2 = 15.06 - 4.15 - 10.00 = 0.91$

7. $3.53X_1 + 1.62X_2 + 2.11X_1^2 + 1.05X_2^2 + 0.50X_1X_2 = 14.75 - 4.10 - 10.00 = 0.65$

8. $3.45X_1 + 1.87X_2 + 1.55X_1^2 + 0.80X_2^2 + 0.44X_1X_2 = 13.94 - 3.30 - 10.00 = 0.64$

9. $3.16X_1 + 1.72X_2 + 2.50X_1^2 + 0.72X_2^2 + 0.75X_1X_2 = 14.67 - 3.39 - 10.00 = 0.68$

10. $3.30X_1 + 1.46X_2 + 2.17X_1^2 + 0.89X_2^2 + 0.60X_1X_2 = 14.61 - 3.35 - 10.00 = 1.26$

"C" Sample Reliability Boundary Equation for Infinite Number of Samples

$$3.24X_1 + 1.69X_2 + 2.13X_1^2 + 0.94X_2^2 + 0.43X_1X_2 = 14.61 - 3.73 - 10.00 = 0.88$$

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RESPONSE SURFACE DETERMINATIONS IN ESTABLISHING THE RELIABILITY OF ONE-SHOT ITEMS, by Nicholas T. Wilburn, February 1964, 72 p. incl. illus. (USAELRDL Technical Report 2428)(DA Task 1G6-22001 A 053-02) Unclassified Report (5 refs)

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2. Response Surface Determination Method

I. Wilburn, Nicholas T.

II. Army Electronics Research and Development Laboratories, Fort Monmouth, N. J.

III. DA Task 1G6 22001 A 053-02

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The response surface determination (RSD) method is proposed for the determination of mean failure points and reliability tolerance limits for a battery design with respect to operational environmental conditions, thermal and dynamic. It provides for the analysis of the battery responses as a function of two or more environmental stresses acting simultaneously, thus affording information on the effects of interactions between forces on the battery performance. Emphasis is placed on providing a maximum of reliability prediction data with small test sample sizes.

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